AD-A258 300

A Low Velocity Approximation for the Relativistic Vlasov-Maxwell System

by

Donald J. McGillen

Submitted in partial fulfillment of the requirements for the degree

Doctor of Philosophy

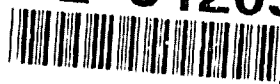Department of Mathematics
Carnegie Mellon
Pittsburgh Pennsylvania

June 1992

92-31209

92 12 09 035

REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE June 1992	3. REPORT TYPE AND DATES COVERED DISSERTATION

4. TITLE AND SUBTITLE
A Low Velocity Approximation for the Relativistic Vlasov-Maxwell System

5. FUNDING NUMBERS

6. AUTHOR(S)

Donald J. McGillen, Lt Col

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

AFIT Student Attending: Carnegie Mellon

8. PERFORMING ORGANIZATION REPORT NUMBER

AFIT/CI/CIA-92-021D

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

AFIT/CI
Wright-Patterson AFB OH 45433-6583

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for Public Release IAW 190-1
Distributed Unlimited
ERNEST A. HAYGOOD, Captain, USAF
Executive Officer

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

14. SUBJECT TERMS

15. NUMBER OF PAGES
102

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT

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Acknowledgements

In loving memory of my father.

Part I

Chapter I: Introduction

The Relativistic Vlasov-Maxwell (RVM) system is a nonlinear system of first order partial differential equations that models the time evolution of a collisionless plasma, e.g. a high temperature, low density ionized gas. Numerical computation of solutions of this system is prohibitively expensive in part because of the six-dimensional phase space for the Vlasov density function. For computational feasibility, we consider a version (RVM) in which the Vlasov density f depends on one spatial variable, x, and two momentum variables, v_1 and v_2. This is the simplest version of the problem which retains the hyperbolic structure of Maxwell's Equations and for which there is a nontrivial magnetic field. We treat the case of a single species of particles with distribution function f, in the presence of a neutralizing background with density n(x). The electric field is given by $E(t,x)=(E_1(t,x), E_2(t,x))$ while the (scalar) magnetic field is denoted $B(t,x)$. The speed of light is c, and we assume that the rest mass and charge of the particles are both 1. The particles move under the action of their self-induced Lorentz force, $F=(E+c^{-1}\hat{v}xB)$, and the objective is to track the simultaneous evolution of the density and the fields, which satisfy the following Cauchy Problem:

(1.1)   (RVM)
$$\begin{cases} \partial_t f + \hat{v}_1\partial_x f + (E+c^{-1}BM\hat{v})\cdot\nabla_v f = 0 \\ \partial_t E_1 = -4\pi j_1 \qquad \partial_x E_1 = 4\pi\rho \\ \partial_t B = -c\partial_x E_2 \qquad \partial_t E_2 = -c\partial_x B - 4\pi j_2 \end{cases}$$

where $\hat{v} = v(1+c^{-2}|v|^2)^{-1/2}$ is the relativistic velocity and $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with Cauchy data

$$f(x,v,0) = f^0(x,v) \geq 0$$

$$E_2(x,0) = E_2^0(x)$$

$$B(x,0) = B^0(x)$$

The charge and current densities, ρ and j, are defined by

$$\rho(t,x) = \int f(t,x,v)dv - n(x)$$

$$j(t,x) = \int \hat{v} f(t,x,v)dv$$

All data functions are taken to be smooth and compactly supported, with $f^0 \geq 0$. The background density, $n(x)$, in addition to being smooth and of compact support, is neutralizing in the sense that

$$\int \rho(0,x)dx = 0$$

We take $E_1(0,x) = 4\pi \int_{-\infty}^{x} \rho(0,y)dy$ as initial data for E_1.

In [2], Glassey and Schaeffer proved global existence of smooth solutions of this problem, their main result being that $f^0 \in C_0^1(\mathbf{R}^3)$ and $E_2^0, B^0 \in C_0^1(\mathbf{R})$ imply the existence of a global, C^1 solution (f,E,B) that satisfies the initial values $f(0,x,v) = f^0(x,v)$, $E^2(0,x) = E_2^0(x)$, $B(0,x) = B^0(x)$ and whose components f, E, and B are compactly supported $\forall \, t > 0$.

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In [3], they proved convergence of a particle-in-cell method for approximating solutions of this system. Their scheme (as well as other typical schemes such as that used in [1]), is limited by the constraint imposed by the Courant-Friedrich-Lewy (CFL) condition on the size of allowable time steps, resulting in enormously expensive computations. CFL requires $\Delta t \leq \Delta x/c$, since the speed of light, c, is the propagation speed for the hyperbolic Maxwell system satisfied by E_2 and B. However, there are regimes in which this restriction is actually far too severe. When the electromagnetic quantities vary slowly, it may be possible to replace Maxwell's equations with simpler models that are elliptic in nature, hence may lead to cheaper computation by allowing larger time steps. (See for example [5].)

In this paper, we propose a modification of (RVM), denoted (RVM~), that employs changes of this nature. The modifications are suggested geometrically by the form of the integral representations of the Maxwell fields E_2 and B which result from solving the 1-D Maxwell system

$$\begin{cases} \partial_t E_2 + c\partial_x B + 4\pi j_2 = 0 \\ \partial_t B + c\partial_x E_2 = 0 \end{cases}$$

The Riemann invariants are $(E_2 + B)$ and $(E_2 - B)$, and we find

(1.2a) $\qquad E_2(t,x) = \frac{1}{2}\left[E_2^0(x-ct)+E_2^0(x+ct)+B^0(x-ct)-B^0(x+ct)\right] -$

$$2\pi \mathcal{E}_+^c[j_2](t,x)$$

(1.2b) $\qquad B(t,x) = \frac{1}{2}\left[E_2^0(x-ct)-E_2^0(x+ct)+B^0(x-ct)+B^0(x+ct)\right] -$

$$2\pi \mathcal{E}_-^c[j_2](t,x)$$

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where $\quad \mathcal{E}_\pm^c [j_2](t,x) := \displaystyle\int_0^t \left[j_2(\tau, x-c(t-\tau)) \pm j_2(\tau, x+c(t-\tau)) \right] d\tau$

Treating the speed of light c as a parameter and allowing it to grow has the effect of "flattening out" the cone integrals appearing in (1.2). Formally, as $c \to \infty$, the cone integrals become purely spatial integrals and we are led to alternate field operators resulting in approximate fields $\tilde{E}_2$ and $\tilde{B}$ given by

(1.3a) $\quad \tilde{E}_2(t,x) = \dfrac{1}{2} \left[E_2^0(x-ct) + E_2^0(x+ct) + B^0(x-ct) - B^0(x+ct) \right] - $

$$2\pi \tilde{\mathcal{E}}_+^c[\tilde{j}_2](t,x)$$

(1.3b) $\quad \tilde{B}(t,x) \;\; = \dfrac{1}{2} \left[E_2^0(x-ct) - E_2^0(x+ct) + B^0(x-ct) + B^0(x+ct) \right] - $

$$2\pi \tilde{\mathcal{E}}_-^c[\tilde{j}_2](t,x)$$

where $\quad \tilde{\mathcal{E}}_\pm^c[\tilde{j}_2](t,x) := \dfrac{1}{c} \displaystyle\int_{-\infty}^x \tilde{j}_2(t,y)dy \;\; \pm \; \dfrac{1}{c} \int_x^\infty \tilde{j}_2(t,y)dy$

and $\quad \tilde{j}(t,x) = \displaystyle\int_{R^2} v\hat{\tilde{f}}(t,x,v)dv \;\;$ is the current density obtained from

the density function $\tilde{f}$ of the modified problem. Note that the same Cauchy data appears in both (1.2) and (1.3), which is natural given that the approximation involved only the integral operators. If we omit the data terms in the expressions for $\tilde{E}_2$ and $\tilde{B}$, what remains, e.g. the integral terms, are solutions of Maxwell's equations as modified with the Darwin and quasi-electrostatic approximations. In the one-space, two-momenta case, this means neglecting the $\partial_t B$ and $\partial_t E_2$ terms in (1.1). However, if we

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make these simplifications of Maxwell first, the representations of the solutions of the resulting system will not contain these data terms. As will be shown later, the presence of these terms results in better (e.g. $1/c^2$ vs $1/c$) convergence of the solutions of the two problems for small time. We propose the following modified version of (RVM):

$$(1.4) \qquad (\text{RVM}^\sim) \begin{cases} \partial_t \tilde{f} + \hat{v}_1 \partial_x \tilde{f} + (\tilde{E} + c^{-1}\tilde{B}M\hat{v}) \cdot \nabla_v \tilde{f} = 0 \\[2mm] 0 \leq f^0(x,v) \in C_0^1(\mathbf{R}^3) \\[2mm] E_2^0(x) \ \& \ B^0(x) \in C_0^1(\mathbf{R}) \end{cases}$$

The data, f^0, E_2^0, and B^0 are considered known. We also assume a smooth, compactly supported, neutralizing background density $n(x)$, as in (RVM). Charge and current densities are given by

$$\tilde{\rho}(t,x) = \int_{\mathbf{R}^2} \tilde{f}(t,x,v)dv - n(x)$$

$$\tilde{j}(t,x) = \int_{\mathbf{R}^2} \hat{v}\tilde{f}(t,x,v)dv$$

The electric and (scalar) magnetic fields are given by

$$\tilde{E}_1(t,x) = 4\pi \int_{-\infty}^{x} \tilde{\rho}(t,y)dy$$

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$$\tilde{E}_2(t,x) = \frac{1}{2}\left[E_2^0(x-ct) + E_2^0(x+ct) + B^0(x-ct) - B^0(x+ct)\right]$$

$$- 2\pi \, \tilde{\mathcal{E}}_+^c[\tilde{j}_2](t,x)$$

$$\tilde{B}(t,x) = \frac{1}{2}\left[E_2^0(x-ct) - E_2^0(x+ct) + B^0(x-ct) + B^0(x+ct)\right]$$

$$- 2\pi \, \tilde{\mathcal{E}}_-^c[\tilde{j}_2](t,x)$$

We assume existence of a global, C^1 solution $(\tilde{f}, \tilde{E}, \tilde{B})$. The proof is straightforward and similar to the existence proof for the 1-D Vlasov Poisson system, once the a-priori bounds obtained in Chapter II are in hand.

In part I of this paper, we will show that solutions of the modified problem converge in a pointwise sense to solutions of the unmodified problem at the asymptotic rate of $1/c^2$. The main result of part I is the following:

Theorem : Let (f,E,B) and $(\tilde{f}, \tilde{E}, \tilde{B})$ be the global, C^1 solutions of (RVM) and (RVM~) respectively, satisfying the same Cauchy data

$$f^0 \in C_0^1(\mathbf{R}^3) \quad \text{and} \quad E_2^0, B^0 \in C_0^1(\mathbf{R}).$$

For c sufficiently large compared to the initial data, there exists a non-decreasing function $D : [\, 0, \infty) \rightarrow [\, 0, \infty)$, depending on the initial data but not on c, such that

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$$\| \, f(t) - \tilde{f}(t) \, \|_\infty \leq \frac{D(t)}{c^2} \quad \text{for all } t > 0$$

Part II of the paper describes numerical experiments designed to corroborate this convergence rate and demonstrate explicitly the value of the modified problem as an approximation to (RVM).

As described in Chapter VI, a particle-in-cell scheme was coded for (RVM~) and output was compared with that obtained using the program from [4]. Besides demonstrating the $1/c^2$ convergence rate of the solutions of the two problems, the results clearly show that the modified problem does not suffer from the CFL limitation - we achieved the same accuracy as the Glassey Schaeffer scheme with significantly larger timesteps.

Additionally, to demonstrate the value of the inclusion of the data terms in the solution of the modified problem, we compared output with a version of the modified problem in which these terms were omitted, i.e. where the expressions for $\tilde{E}_2$ and $\tilde{B}$ are solutions of the system obtained by making the Darwin and quasi-electrostatic modifications of Maxwell's equations. We found that including the data terms improved the accuracy of the solutions with very little increase in computation time. These results are discussed in more detail in Chapter VIII.

Throughout this paper, we will use the following conventions and notation:

    ·c is the speed of light. We always assume $c \geq 1$.

    ·D will denote a generic positive constant which depends on the initial data, but not on c.

- D(t) will denote a positive, non-decreasing function with domain $[0,\infty)$.

- Partial derivatives will be denoted by subscripts. (for example $\partial_x = \dfrac{\partial}{\partial x}$)

- We will frequently abbbreviate the arguments in the expressions for the characteristics, writing for example $X(s)$ and $V(s)$ for $X(s,t,x,v)$ and $V(s,t,x,v)$.

- We will write $f(t)$ for $f(t,\cdot,\cdot)$ and $f(t,x)$ for $f(t,x,\cdot)$.

- Finally, we will use the following norms:

For a scalar-valued function $g=g(t,x,v)$,

$$\|g(t)\|_\infty = \sup\{\,|g(t,x,v)|\ \text{such that}\ x\in\mathbf{R}\ \text{and}\ v\in\mathbf{R}^2\}$$

$$\|g\|_\infty = \sup\{\,|g(t,x,v)|\ \text{such that}\ t>0,\ x\in\mathbf{R}\ \text{and}\ v\in\mathbf{R}^2\}$$

$$\|g(t)\|_{c^1} = \|g(t)\|_\infty + \|(\partial_x g)(t)\|_\infty + \|(\nabla_v g)(t)\|_\infty$$
$$= \|g(t)\|_\infty + \|(\partial_x g)(t)\|_\infty + \|(\partial_{v_1} g)(t)\|_\infty + \|(\partial_{v_2} g)(t)\|_\infty$$

For a vector-valued function $H = H(t,x) = (H_1(t,x), H_2(t,x))$,

$$\|H(t)\|_\infty = \|H_1(t)\|_\infty + \|H_2(t)\|_\infty$$

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Chapter II : A-priori Bounds for the The Unmodified Problem

In their proof of global existence of smooth solutions for (RVM) with c = 1, Glassey and Schaeffer used a standard iteration scheme, convergence of the iterates being assured once a-priori C^1 bounds on f, E, and B are known. For the purposes of this paper, we need corresponding bounds for the case where c appears as a parameter (c > 1). In particular, we must determine which bounds can be taken to be independent of c. We assume, for any fixed c > 1, existence of a global-in-time C^1 solution (f, E, B), compactly supported $\forall$ t > 0. Proof of existence is omitted, since the argument would be essentially identical to that in [2], using the a-priori bounds we will find in this chapter, which are obtained following the methods in [2].

A. Density Estimates

Define X(s,t,x,v) and V(s,t,x,v) by

(2.1a)
$$\begin{cases} \dfrac{d}{ds} X(s,t,x,v) = \hat{V}_1(s,t,x,v) \\ X(t,t,x,v) = x \end{cases}$$

(2.1b)
$$\begin{cases} \dfrac{d}{ds} V(s,t,x,v) = E(s,X(s))+c^{-1}B(s,X(s))M\hat{V}(s) \\ V(t,t,x,v) = v \end{cases}$$

Here, X(s) and V(s) abbreviate, respectively, X(s,t,x,v) and V(s,t,x,v). We immediately have a uniform bound on f(t,x,v), since

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$$\frac{d}{ds} f(s, X(s,t,x,v), V(s,t,x,v)) = 0 \quad \text{by Vlasov, so}$$

$$
\begin{aligned}
f(t,x,v) &= f(t,X(t,t,x,v),\ V(t,t,xv)) \\
&= f(s,X(s,t,x,v),\ V(s,t,x,v)) \\
&= f(0,X(0,t,x,v),\ V(0,t,x,v)) \\
&= f^0(X(0,t,x,v),\ V(0,t,x,v))
\end{aligned}
$$

$$\Rightarrow \qquad \sup \{ f(t,x,v) : x \in \mathbf{R}, v \in \mathbf{R}^2 \} = \| f^0 \|_\infty$$

B. Charge Conservation

Lemma [2.1]: $\iint f(t,x,v)dvdx = \iint f^0(x,v)dvdx$

Proof :   Integrating the Vlasov equation in v yields

$$\int \partial_t f dv + \int \hat{v}_1 \partial_x f dv + \int [(E+c^{-1}BM\hat{v})\cdot\nabla_v f]\, dv = 0$$

or $\qquad \partial_t\rho + \partial_x j_1 + \int[(E+c^{-1}BM\hat{v})\cdot\nabla_v f]dv = 0$

Lemma [2.2]: $(E+c^{-1}BM\hat{v})\cdot\nabla_v f = \nabla_v\cdot[(E+c^{-1}BM\hat{v})f]$

Proof: $\nabla_v\cdot[(E+c^{-1}BM\hat{v})f] = \partial_{v_1}[(E_1+c^{-1}B\hat{v}_2)f] + \partial_{v_2}[(E_2-c^{-1}B\hat{v}_1)f]$

$$= (E_1+c^{-1}B\hat{v}_2)\partial_{v_1}f + c^{-1}fB\partial_{v_1}\hat{v}_2$$
$$+ (E_2-c^{-1}B\hat{v}_1)\partial_{v_2}f - c^{-1}fB\partial_{v_2}\hat{v}_1$$

$$= (E+c^{-1}BM\hat{v})\cdot\nabla_v f + c^{-1}fB(\partial_{v_1}\hat{v}_2 - \partial_{v_2}\hat{v}_1)$$

But $\qquad \partial_{v_1}\hat{v}_2 - \partial_{v_2}\hat{v}_1 = \partial_{v_1}\left(\dfrac{v_2}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}}\right) - \partial_{v_2}\left(\dfrac{v_1}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}}\right)$

= 0 and the claim is verified.

Using the compact support of f and the divergence theorem then yields

(2.2) $\partial_t \rho + \partial_x j_1 = 0$

Now j_1 is compactly supported in x $\forall t > 0$, as follows from the definition of j and the compact support of f. Integrating (2.2) over x then gives

$$\partial_t \int \rho(t,x)dx = 0$$

$$\Rightarrow \qquad \partial_t \int [\int f(t,x,v)dv - n(x)]dx = 0$$

$$\Rightarrow \qquad \int [\int f(t,x,v)dv - n(x)]dx = const$$

$$\Rightarrow \qquad \int [\int f(t,x,v)dv - n(x)]dx = \int [\int f(0,x,v)dv - n(x)]dx$$

$$= \int [\int f^0(x,v)dv - n(x)]dx$$

$$\Rightarrow \qquad \iint f(t,x,v)dvdx = \iint f^0(x,v)dvdx \qquad \forall \, t > 0, \quad \text{which proves the}$$

lemma, and establishes global-in-time charge conservation.

C. Field Estimates

(1) Uniform Bound for E_1

We first establish that $E_1(t,x)$ is uniformly bounded $\forall t > 0$ and $\forall x$, with a bound that is independent of c. Integrating Gauss' Law for E_1 with respect to x, we get

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$$E_1(t,x) = 4\pi \int_{-\infty}^{x} \rho(t,y)dy + \text{const}$$

and hence

$$E_1(0,x) = 4\pi \int_{-\infty}^{x} \rho(0,y)dy$$

since the constant has been chosen to be 0 already. (the only choice resulting in a finite energy solution). Therefore,

$$E_1(t,x) = 4\pi \int_{-\infty}^{x} \rho(t,y)dy$$

$$= 4\pi \int_{-\infty}^{x} [\int f(t,y,v)dv - n(y)]dy$$

Using the non-negativity of f and lemma [2.1], we have

$$|E_1(t,x)| \leq 4\pi\iint f(t,y,v)dvdy + 4\pi\int |n(y)| \, dy$$

$$= 4\pi\iint f^0(y,v)dvdy + 4\pi\int |n(y)| \, dy$$

$$\Rightarrow \quad \sup_{x, \; t>0} |E_1(t,x)| \leq D,$$

by assumptions on the data, and we have a bound on E_1 that is independent of t, x, and c.

(2) <u>Compact Support of $E_1(0,x)$</u>

Let ξ be chosen s.t. $f^0(x,v) = 0$ and $n(x) = 0 \; \forall x$ s.t. $|x| > \xi$, and consider

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$$E_1(0,x) = 4\pi \int_{-\infty}^{x} [\int f^0(y,v)dv - n(y)]dy$$

Suppose $x > \xi$. Then

$$4\pi \int_{-\infty}^{x} \left(\int f^0(y,v)dv - n(y)\right)dy = \iint f^0(y,v)dvdy - \int n(y)dy = 0$$

by the assumption of global neutrality. If $x < -\xi$,

$$\int_{-\infty}^{x}\int f^0(y,v)dvdy = 0 = \int_{-\infty}^{x} n(y)dy$$

$$\Rightarrow \qquad E_1(0,x) = 0 \text{ for } |x| > \xi$$

(3) Uniform Bounds for E_2 and B

We employ integral representations

(2.3a) $\qquad E_2(t,x) = \dfrac{1}{2} [E_2^0(x-ct)+E_2^0(x+ct)+B^0(x-ct)-B^0(x+ct)]$

$$-4\pi \int_{0}^{t} [j_2(\tau,x-c(t-\tau))+j_2(\tau,x+c(t-\tau))]d\tau$$

(2.3b) $\qquad B(t,x) = \dfrac{1}{2} [E_2^0(x-ct)-E_2^0(x+ct)+B^0(x-ct)+B^0(x+ct)$

$$-4\pi \int_{0}^{t} [j_2(\tau,x-c(t-\tau))-j_2(\tau,x+c(t-\tau))]d\tau$$

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By hypothesis, the data terms are uniformly bounded independently of c, so it suffices to show that

$$\sup_{x\ t>0}\ \left|\ \int_0^t j_2(\tau,x\pm c(t-\tau))d\tau\ \right|\quad\text{is uniformly bounded.}$$

By definition, $j_2(t,x) = \int \hat{v}_2 f(t,x,v)dv = \int \dfrac{v_2}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}} f(t,x,v)dv$

$$\Rightarrow\quad |j_{2(t,x)}| \le \int \dfrac{|v_2|}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}} f(t,x,v)dv$$

Lemma [2.3]:   There exists a constant D, independent of c and depending only on the data, such that

$$\sup_{x\ t>0} \int_0^t \int \dfrac{|v_2|}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}} f(\tau,x\pm c(t-\tau),v)dvd\tau \le D$$

Corollary [2.3]:   The fields E_2 and B are uniformly bounded :   There exists a constant D, independent of c, such that

$$\| E_2(t) \|_\infty + \| B(t) \|_\infty \le D \text{ for all } t > 0$$

Proof of the Corollary :   By the integral representations of E_2 and B, we

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need to bound

$$\sup_{x\, t\, >\, 0}\ \left|\ \int_0^t j_2(\tau, x \pm c(t-\tau))d\tau\ \right|$$

$$\leq \sup_{x\, t\, >\, 0}\ \int_0^t \int \frac{|v_2|}{\sqrt{1+c^{-2}(v_1^2+v_2^2)}} f(\tau, x \pm c(t-\tau),\, v)dvd\tau$$

$$\leq\ D \quad \text{by the lemma.}$$

Proof of the lemma : We first derive an energy identity for Vlasov. The relativistic energy of a particle is $c^2\sqrt{1+c^{-2}|v|^2}$, so the energy density is given by

$$e_k = \int c^2\sqrt{1+c^{-2}|v|^2}\ f(t,x,v)dv$$

Letting $\gamma = \sqrt{1+c^{-2}|v|^2}$ and differentiating with respect to t, we have by Vlasov and lemma [2.2],

$$\partial_t e_k = \int c^2\, \gamma\, \partial_t f\ dv$$

$$= -\int c^2\, \gamma\, [\ \hat{v}_1 \partial_x f + (E+c^{-1}BM\hat{v}) \cdot \nabla_v f\ ]dv$$

$$= -\partial_x \int c^2 v_1 f dv - \int c^2\gamma\, [\ (E+c^{-1}BM\hat{v}) \cdot \nabla_v f\ ]dv$$

$$= -\partial_x \int c^2 v_1 f dv - \int c^2\gamma\, \nabla_v \cdot [\ (E+c^{-1}BM\hat{v})f\ ]dv$$

Integrating by parts and using compact support of f, we have

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$$\partial_t e_k \;=\; -\,\partial_x \int c^2 v_1 f\,dv \;+\; c^2 \int (\,[\,(E+c^{-1}BM\hat{v})f\,]\cdot\nabla_v\gamma)\,dv$$

$$=\; -\,\partial_x \int c^2 v_1 f\,dv \;+\; c^2 \int [\,(E+c^{-1}BM\hat{v})f\,]\cdot\frac{\hat{v}}{c^2}\,dv$$

$$=\; -\,\partial_x \int c^2 v_1 f\,dv \;+\; \int (\,[\,(E+c^{-1}BM\hat{v})f\,]\cdot\hat{v})\,dv$$

Now, since $(\hat{v}\times B)\cdot\hat{v}=0$,

$$(E+c^{-1}BM\hat{v})f\cdot\hat{v} \;=\; (E+c^{-1}(\hat{v}\times B))f\cdot\hat{v}$$
$$=\; Ef\cdot\hat{v}$$
$$=\; E\cdot f\hat{v}$$

$$\Rightarrow\qquad \int (\,[\,(E+c^{-1}BM\hat{v})f\,]\cdot\hat{v})\,dv \;=\; \int E\cdot f\hat{v}\,dv$$
$$=\; E\cdot\left(\int f\hat{v}\,dv\right)$$
$$=\; E\cdot j$$

so $\qquad \partial_t e_k = -\partial_x \int c^2 v_1 f\,dv + E\cdot j$

From Maxwell,

$$-4\pi j_1 = \partial_t E_1 \quad\text{and}\quad -4\pi j_2 = \partial_t E_2 + c\partial_x B, \quad\text{so}$$

$$j = -\frac{1}{4\pi}\,(\partial_t E_1,\; \partial_t E_2 + c\partial_x B)$$

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$$\Rightarrow \qquad E \cdot j \;=\; -\frac{1}{4\pi}\left(E_1\partial_t E_1 + E_2\partial_t E_2 + cE_2\partial_x B\right)$$

$$=\; -\frac{1}{4\pi}\left[\frac{1}{2}\frac{\partial}{\partial t}\left(\,|E_1|^2 + |E_2|^2\,\right) + cE_2\partial_x B\right]$$

$$=\; -\frac{1}{8\pi}\frac{\partial}{\partial t}|E|^2 - \frac{1}{4\pi}\left(cE_2\partial_x B + c(\partial_x E_2)B - c(\partial_x E_2)B\right)$$

$$(2.4)\qquad =\; -\frac{1}{8\pi}\frac{\partial}{\partial t}|E|^2 - \frac{1}{4\pi}\frac{\partial}{\partial x}(cE_2 B) + \frac{1}{4\pi}c(\partial_x E_2)B$$

But again from Maxwell, $\partial_x E_2 = -\dfrac{1}{c}\partial_t B,$ so we can write the last term in (2.4) as

$$\frac{1}{4\pi}c(\partial_x E_2)B \;=\; -\frac{1}{4\pi}c\left(\frac{1}{c}\partial_t B\right)B \;=\; -\frac{1}{4\pi}B\,\partial_t B$$

$$=\; -\frac{1}{4\pi}\cdot\frac{1}{2}\frac{\partial}{\partial t}|B|^2 \;=\; -\frac{1}{8\pi}\frac{\partial}{\partial t}|B|^2$$

$$\Rightarrow \qquad E\cdot j \;=\; -\frac{1}{8\pi}\frac{\partial}{\partial t}\left(|E|^2 + |B|^2\right) - \frac{1}{4\pi}\frac{\partial}{\partial x}(cE_2 B)$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}e_k \;=\; -\frac{\partial}{\partial x}\left(c^2\int v_1 f\,dv + \frac{c}{4\pi}E_2 B\right) - \frac{1}{8\pi}\frac{\partial}{\partial t}\left(|E|^2 + |B|^2\right)$$

Regrouping terms gives

$$(2.5)\qquad \frac{\partial}{\partial t}\left[\,4\pi e_k + \frac{1}{2}\left(|E|^2 + |B|^2\right)\right] \;=\; -\frac{\partial}{\partial x}\left(4\pi c^2\int v_1 f\,dv + cE_2 B\right)$$

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Let $\quad e = 4\pi e_k + \frac{1}{2}(|E|^2 + |B|^2) = 4\pi c^2 \int \sqrt{1+c^{-2}|v|^2}\,fdv + \frac{1}{2}(|E|^2 + |B|^2)$

and $\quad m = -4\pi c^2 \int v_1 fdv - cE_2B$

Then by (2.5),

$$\frac{\partial e}{\partial t} = \frac{\partial m}{\partial x}$$

which is the energy identity we seek. Continuing with the proof of lemma [2.3], we integrate

$$0 = \frac{\partial e}{\partial t} - \frac{\partial m}{\partial x}$$

over a backwards characteristic cone with vertex (x,t) and interior T, using Green's theorem:

$$0 = \iint_T \left[\frac{\partial e}{\partial t} - \frac{\partial m}{\partial x}\right]dA = \iint_T \nabla_{x,t} \cdot (-m,e)dA$$

$$= \int_{\partial T} (-m,e)\cdot\hat{n}ds$$

$$= \int_{x-ct}^{x+ct} (-m,e)\Big|_{(y,0)} \cdot (0,-1)dy$$

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$$+ \int_0^t [(-m,e)]\big|_{(\tau,x+c(t-\tau))} \cdot \frac{1}{\sqrt{1+c^2}} (1,c)(\sqrt{1+c^2})]d\tau$$

$$+ \int_t^0 [(-m,e)]\big|_{(\tau,x-c(t-\tau))} \cdot \frac{1}{\sqrt{1+c^2}} (-1,c)(-\sqrt{1+c^2})]d\tau$$

which reduces to

(2.6)
$$\int_{x-ct}^{x+ct} e(y,o)dy = \int_0^t [(ce-m)(\tau,x+c(t-\tau)) + (ce+m)(\tau,x-c(t-\tau))]d\tau$$

Claim : There exists a constant D, depending only on the data, such that for all $c \geq 1$,

$$\int_{x-ct}^{x+ct} e(y,o)dy \leq Dc^2$$

Proof of the claim : By definition,

$$e(y,0) = \frac{1}{2}\left[(E_1^0(y))^2 + (E_2^0(y))^2 + (B^0(y))^2\right] +$$

$$4\pi c^2 \int \sqrt{1+c^{-2}|v|^2} \, f^0(y,v)dv)$$

E_2^0, B^0, and f^0 are smooth and compactly supported by hypothesis, while E_1^0 is compactly supported as shown earlier. Hence $e(y,0)$ is compactly

supported in y , and

$$\int_{x-ct}^{x+ct} e(y,0)dy = \int_{x-ct}^{x+ct} \frac{1}{2}\left[(E_1^0(y))^2 + (E_2^0(y))^2 + (B^0(y))^2\right]dy +$$

$$4\pi c^2 \int_{x-ct}^{x+ct}\int \sqrt{1+c^{-2}|v|^2}\, f^0(y,0))dvdy$$

$$\leq D + Dc^2$$

$$\leq Dc^2 \text{ for } c \geq 1$$

and so by (2.6) we have

$$(2.7) \qquad \int_0^t [(ce-m)(\tau,x+c(t-\tau)) + (ce+m)(\tau,x-c(t-\tau))]d\tau \leq Dc^2$$

Now $\qquad ce \pm m = \dfrac{c}{2}|E_1|^2 + \dfrac{c}{2}(E_2 \mp B)^2 + 4\pi c^2\int[c\sqrt{1+c^{-2}|v|^2} \mp v_1]f(t,x,v)$

We consider the integrand in the last term:

$$c\sqrt{1+c^{-2}|v|^2} \mp v_1 = \frac{(c\sqrt{1+c^{-2}|v|^2} \mp v_1)\,(c\sqrt{1+c^{-2}|v|^2} \pm v_1)}{(c\sqrt{1+c^{-2}|v|^2} \pm v_1)}$$

$$= \frac{c^2 + v_2^2}{c\sqrt{1+c^{-2}|v|^2} \pm v_1} \geq \frac{c^2 + v_2^2}{c\sqrt{1+c^{-2}|v|^2} + |v_1|}$$

$$\geq \frac{c^2 + v_2^2}{2c\sqrt{1+c^{-2}|v|^2}}$$

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Note that $(|v_2| - c)^2 \geq 0 \Rightarrow |v_2|^2 - 2c|v_2| + c^2 \geq 0$ and so

$$\frac{v_2^2 + c^2}{2c} \geq |v_2|$$

$$\Rightarrow \quad \frac{c^2 + v_2^2}{2c\sqrt{1+c^{-2}|v|^2}} \geq \frac{|v_2|}{\sqrt{1+c^{-2}|v|^2}}$$

which implies

$$(2.8) \qquad c\sqrt{1+c^{-2}|v|^2} + v_1 \geq \frac{|v_2|}{\sqrt{1+c^{-2}|v|^2}}$$

Using (2.6), (2.7) and (2.8) and discarding the field terms, since they are positive, we have

$$c^2 \int_0^t \int \left[c\sqrt{1+c^{-2}|v|^2} + v_1 \right] f(\tau, x \pm c(t-\tau)), v)\,dv\,d\tau \leq Dc^2$$

$$\Rightarrow \quad c^2 \int_0^t \int \frac{|v_2|}{\sqrt{1+c^{-2}|v|^2}} f(\tau, x \pm c(t-\tau), v)\,dv\,d\tau \leq Dc^2$$

$$\Rightarrow \quad \sup_{x\, t > 0} \int_0^t \int \frac{|v_2|}{\sqrt{1+c^{-2}|v|^2}} f(\tau, x \pm c(t-\tau), v)\,dv\,d\tau \leq D$$

for all x and for all t > 0, where D is independent of c. This proves lemma [2.3].

Summary : The fields $E_1(t,x)$, $E_2(t,x)$, and $B(t,x)$ are uniformly bounded for all x and for all $t > 0$. The bound is independent of c and depends only on the data. These field bounds enable us to obtain bounds on the x-support and the v-support of f.

D. **Bounds on the Support of f**

(1) v - support: Let $P_1(t) = \sup \{ |v| : \exists \ x \ \text{s.t.} \ f(t,x,v) \neq 0 \}$

Lemma [2.4]: There exists a constant D, independent of c, such that

$$P_1(t) \leq D(1+t).$$

Proof :   Integrating the characteristic equation for V gives

$$V(0,t,x,v) - v = \int_t^0 \left[E(s,X(s)) + c^{-1}B(s,X(s))M\hat{V}(s) \right] ds$$

$$\Rightarrow \qquad v = V(0,t,x,v) + \int_0^t \left[E(s,X(s)) + c^{-1}B(s,X(s))M\hat{V}(s) \right] ds$$

By the assumption that $f^0 \in C_0^1(\mathbf{R}^3)$, there exists D s.t.

$$f(t,x,v) = f^0(X(0,t,x,v), V(0,t,x,v)) = 0 \ \text{whenever} \ |V(0,t,x,v)| \geq D.$$

Then, since $\dfrac{|\hat{V}|}{c} \leq 1$, we may write

(2.9) $\qquad |v| \leq D + \int_0^t (\|E\|_\infty + \|B\|_\infty) ds \qquad$ on the support of $f(t,\cdot,\cdot)$.

$$\leq D \ (1+t) \ \text{by the uniform field bounds.}$$

(2) x - support: Let $P_2(t) = \sup \{ |x|: \exists$ v s.t. $f(t,x,v) \neq 0 \}$

Lemma [2.5]:   There exists a constant D, independent of c, such that

$$P_2(t) \leq D(1+t)^2$$

Proof: Integrating the characteristic equation for X gives

$$x = X(0,t,x,v) + \int_0^t \frac{V_1(s)}{\sqrt{1+c^{-2}|V(s)|^2}} \, d s$$

As in (2.9), using the compact support of f^0, we may write

$$|x| \leq D + \int_0^t \frac{|V_1(s)|}{\sqrt{1+c^{-2}|V(s)|^2}} \, ds \quad \text{on the support of } f(t,\cdot,\cdot)$$

$$\leq D + \int_0^t |V_1(s)| \, ds$$

By lemma [2.4], we have

$$|x| \leq D + \int_0^t D(1+t)ds \leq D(1+t)^2 \quad \text{on the support of } f(t,\cdot,\cdot).$$

E. Bounds on ρ and j

Corollary [2.4]: $|\rho(t,x)| + |j(t,x)| \leq D(1+t)^3$ for all x and for all $t > 0$,

where D is again a constant independent of t, x, and c

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and depending only on the data.

Proof :   From the definition of   j(t,x),   we have immediately:

$$|j(t,x)| \leq \int |\hat{v}|f(t,x,v)dv$$

$$\leq \int |v|f(t,x,v)dv$$

$$\leq \int_{v\text{-spt of } f} D(1+t)dv$$

$$\leq D(1+t)^3 \quad \text{for all } x \text{ and for all } t \geq 0$$

For ρ, we have

$$|\rho(t,x)| \leq \int |f(t,x,v)|dv + |n(x)|$$

$$\leq D(1+t)^2 + D$$

$$\leq D(1+t)^2 \quad \forall \ x, \ t > 0,$$

and the corollary is proved.

F. **Estimates on Derivatives**

We are now able to bound the derivatives of   f(t),   E(t,x)   and   B(t,x),
which will give us bounds on C^1 norms.

Lemma [2.6]:   With the given assumptions on the Cauchy data,   there
exists   a non-decreasing function   D: $(0,\infty) \rightarrow (0,\infty)$ which is

independent of c, such that

$$\|f(t)\|_{c^1} + \|E(t)\|_{c^1} + \|B(t)\|_{c^1} \leq D(t)$$

Proof : We begin with bounds on the spatial derivatives of E and B.

(1) <u>Spatial Derivative of E_1</u>: Using Maxwell, assumptions on n and non negativity of f, we have

$$\partial_x E_1(t,x) \;=\; 4\pi\rho(t,x) \;=\; 4\pi \left(\int f(t,x,v)dv \;-\; n(x) \right)$$

$$\Rightarrow \qquad |\partial_x E_1(t,x)| \;\leq\; 4\pi \int fdv + D$$

$$\leq D \cdot (\text{radius of v-support of } f(t,x))^2 + D$$

$$\leq D(t) \quad \text{by lemma [2.4]}$$

(2) <u>Spatial Derivatives of E_2 and B</u>:

To obtain bounds on $\partial_x E_2$ and $\partial_x B$, we set

$$K^{\pm}(t,x) \;=\; E_2(t,x) \;\pm\; B(t,x)$$

From the integral representations (2.3), we have

$$(2.10) \qquad K^{\pm}(t,x) \;=\; K^{\pm,0}(x \mp ct) \;-\; 4\pi \int_0^t j_2(\tau, x \mp c(t-\tau))d\tau$$

where $\qquad K^{\pm,0}(x \mp ct) \;=\; E_2^0(x \mp ct) \;\pm\; B^0(x \mp ct)$

We estimate $\partial_x K^+$ only, since the bounds for K^- are obtained in a similar way. Differentiating (2.10) with respect to x and using the definition of j_2, we have

$$(2.11) \qquad \partial_x K^+(t,x) = \partial_x K^{+,0}(x-ct) - 4\pi \int_0^t \partial_x \int \hat{v}_2 f(\tau, x-c(t-\tau), v)\, dv\, d\tau$$

Now $\quad |\partial_x K^{+,0}(x-ct)| = |\partial_x E_2^0(x-ct) + \partial_x B^0(x-ct)| \le D$, since E_2^0, $B^0 \in C_0^2 (\mathbf{R})$.

As in [2], we introduce the following differential operators:

$$T_+ := \partial_t + c\partial_x \quad \text{This is the derivative along one of the}$$

characteristics of the Maxwell system, e.g.

$$T_+ f(\tau, x-c(t-\tau), v) = \frac{\partial}{\partial \tau} f(\tau, x-c(t-\tau), v)$$

$$S := \partial_t + \hat{v}_1 \partial_x$$

Note that by Vlasov and Lemma [2.2],

$$Sf(\tau, x-c(t-\tau), v) = (\partial_t + \hat{v}_1 \partial_x) f(\tau, x-c(t-\tau), v)$$

$$= -((E+c^{-1}BM\hat{v}) \cdot \nabla_v f)\big|_{(\tau, x-c(t-\tau), v)}$$

$$= -(\nabla_v \cdot (E+c^{-1}BM\hat{v})f)\big|_{(\tau, x-c(t-\tau), v)}$$

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These can be inverted to obtain

$$\partial_t = \frac{cS - \hat{v}_1 T}{c - \hat{v}_1} \quad \text{and} \quad \partial_x = \frac{T_+ - S}{c - \hat{v}_1}$$

Replacing ∂_x in (2.11), we have

$$\partial_x K^+ = \partial_x K^{+,0}(x-ct) - 4\pi \int_0^t \iint \frac{\hat{v}_2}{c - \hat{v}_1}(T_+ - S)f(\tau, x - c(t-\tau), v)dvd\tau$$

$$= \partial_x K^{+,0}(x-ct) - 4\pi \int_0^t \iint \frac{\hat{v}_2}{c - \hat{v}_1}\left[\frac{\partial f}{\partial \tau}(\tau, x - c(t-\tau), v) - Sf(\tau, x - c(t-\tau), v)\right]dvd\tau$$

$$= \partial_x K^{+,0}(x-ct) - 4\pi \int_0^t \frac{\partial}{\partial \tau}\int \frac{\hat{v}_2}{c - \hat{v}_1} f(\tau, x - c(t-\tau), v)dvd\tau$$

$$- 4\pi \int_0^t \iint \frac{\hat{v}_2}{c - \hat{v}_1} \nabla_v \cdot ((E + c^{-1}BM\hat{v})f)\bigg|_{(\tau, x - c(t-\tau), v)} dvd\tau$$

Since $f(t, \cdot, \cdot)$ is compactly supported, $c - \hat{v}_1 \neq 0$, and the v-integrals are non-singular.

$$\Rightarrow \quad \partial_x K^+ = \partial_x K^{+,0}(x-ct) - 4\pi \int \frac{\hat{v}_2}{c - \hat{v}_1} f(t, x, v)dv + 4\pi \int \frac{\hat{v}_2}{c - \hat{v}_1} f(0, x-ct, v)dv$$

$$- 4\pi \int_0^t \iint \frac{\hat{v}_2}{c - \hat{v}_1} \nabla_v \cdot ((E + c^{-1}BM\hat{v})f)\bigg|_{(\tau, x - c(t-\tau), v)} dvd\tau$$

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We integrate the last term and use the compact v support of f to get

$$\int_0^t \int \frac{\hat{v}_2}{c-\hat{v}_1} \nabla \cdot ((E+c^{-1}BM\hat{v})f)\Big|_{(\tau,x-c(t-\tau),v)} dvd\tau \ =$$

$$0 \ - \ \int_0^t \int \left((E+c^{-1}BM\hat{v})f\Big|_{(\tau,x-c(t-\tau),v)}\right) \cdot \nabla_v \left(\frac{\hat{v}_2}{c-\hat{v}_1}\right) dvd\tau$$

So we have

$$\partial_x K^+ \ = \ \partial_x K^{+,0}(x-ct) \ - \ 4\pi \int \frac{\hat{v}_2}{c-\hat{v}_1} f(t,x,v)dv) \ + \ 4\pi \int \frac{\hat{v}_2}{c-\hat{v}_1} f(0,x-ct,v)dv$$

$$+ \ 4\pi \int_0^t \int \nabla_v \left(\frac{\hat{v}_2}{c-\hat{v}_1}\right) \cdot ((E+c^{-1}BM\hat{v})f)\Big|_{(\tau,x-c(t-\tau),v)} dvd\tau$$

which implies

$$(2.12) \qquad |\partial_x K^+(t,x)| \ \leq \ D + 4\pi \int \frac{|\hat{v}_2|}{c-\hat{v}_1} f(t,x,v)dv \ + \ 4\pi \int \frac{|\hat{v}_2|}{c-\hat{v}_1} f^0(x-ct,v)dv$$

$$+ \ 4\pi \int_0^t \int \left\|\nabla_v \left(\frac{\hat{v}_2}{c-\hat{v}_1}\right)\right\| \ \|E+c^{-1}BM\hat{v}\| \ |f(\tau,x-c(t-\tau),v)|dvd\tau$$

since the spatial derivatives of E_2^0 and B^0 are uniformly bounded.

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The v-integrals are taken over a compact set, e.g. $\{v: |v| \leq D(1+t)\}$, so $c - \hat{v}_1 \neq 0$, and

(2.13) $\dfrac{\hat{v}_2}{c - \hat{v}_1} \leq D(t) \quad \forall\, t > 0$

Also, we have

$$\nabla_v \left(\frac{\hat{v}_2}{c - \hat{v}_1} \right) = \frac{1}{c - \hat{v}_1} \nabla_v \hat{v}_2 + \frac{\hat{v}_2}{(c - \hat{v}_1)^2} \nabla_v \hat{v}_1$$

and so

(2.14) $\left\| \nabla_v \left(\dfrac{\hat{v}_2}{c - \hat{v}_1} \right) \right\| \leq \dfrac{1}{|c - \hat{v}_1|} \| \nabla_v \hat{v}_2 \| + \dfrac{|\hat{v}_2|}{(c - \hat{v}_1)^2} \| \nabla_v \hat{v}_1 \|$

Lemma [2.7]: $\| \nabla_v \hat{v}_i \| \leq \sqrt{2}$, $i = 1, 2$

Proof:

$$\| \nabla_v \hat{v}_i \|^2 = \left(\frac{1}{\sqrt{1 + c^{-2}|v|^2}} - \left(\frac{v_i}{c} \right)^2 \frac{1}{\left(1 + c^{-2}|v|^2 \right)^{3/2}} \right)^2$$

$$+ \frac{(v_i v_j)^2}{c^4 (1 + c^{-2}|v|^2)^3} \qquad i = 1, 2;\ j = \{1, 2\}\backslash i$$

Now $\dfrac{1}{\sqrt{1 + c^{-2}|v|^2}} - \left(\dfrac{v_i}{c} \right)^2 \dfrac{1}{\left(1 + c^{-2}|v|^2 \right)^{3/2}} = \dfrac{1 + c^{-2} v_j^2}{\left(1 + c^{-2}|v|^2 \right)^{3/2}} \geq 0$, so

$$\left(\frac{1}{\sqrt{1 + c^{-2}|v|^2}} - \left(\frac{v_i}{c} \right)^2 \frac{1}{\left(1 + c^{-2}|v|^2 \right)^{3/2}} \right)^2 \leq \left(\frac{1}{\sqrt{1 + c^{-2}|v|^2}} \right)^2 = \frac{1}{1 + c^{-2}|v|^2}$$

so
$$\|\nabla_v \hat{v}_i\| \le \left(\frac{1}{1+c^{-2}|v|^2} + \frac{(v_1 v_2)^2}{c^4(1+c^{-2}|v|^2)^3} \right)^{1/2}$$

$$\le \sqrt{2} \quad \text{and the lemma is proved.}$$

Then from (2.13) and (2.14) we have

$$(2.15) \qquad \left\| \nabla_v \left(\frac{\hat{v}_2}{c-\hat{v}_1} \right) \right\| \le D(t)$$

Using (2.12), (2.15), and the bounds on f, f^0, and the fields then gives

$$\| \partial_x K^+(t) \|_\infty \le D(t) \quad \text{for all } t > 0, \quad D(t) \text{ independent of } x \text{ and } c.$$

A similar treatment (see remark on page 16 of [2]) yields

$$\|\partial_x K^-(t)\|_\infty \le D(t), \text{ and the result is that}$$
$$\|\partial_x K(t)\|_\infty \le D(t) \quad \forall \, t > 0$$

Using the bounds on the fields themselves, we have

$$\|E_2(t)\|_{c^1}, \quad \|B(t)\|_{c^1} \le D(t)$$

(3) <u>Derivatives of Characteristics:</u>
The next step is to bound ∂X, ∂V_1 and ∂V_2, where ∂ can be ∂_x, ∂_{v_1}, or ∂_{v_2}.

Integrating (2.1) gives

$$X(s,t,x,v) = x - \int_s^t \hat{V}_1(\xi,t,x,v)d\xi$$

$$V_1(s,t,x,v) = v_1 - \int_s^t [E_1(\xi,X,(\xi))+c^{-1}B(\xi,X(\xi))\hat{V}_2(\xi)]d\xi$$

$$V_2(s,t,x,v) = v_2 - \int_s^t [E_2(\xi,X(\xi))-c^{-1}B(\xi,X(\xi))\hat{V}_1(\xi)]d\xi$$

We first consider the spatial derivatives. Differentiating with respect to x, we find

$$\frac{\partial X(s)}{\partial x} = 1 - \int_s^t \partial_x \hat{V}_1(\xi)d\xi$$

$$\frac{\partial V_1(s)}{\partial x} = -\int_s^t \Big[\partial_x E_1(\xi,X(\xi))\frac{\partial X}{\partial x} +$$

$$c^{-1}\Big\{B(\xi,X(\xi))\partial_x\hat{V}_2(\xi) + \partial_x B(\xi,X(\xi))\frac{\partial X(\xi)}{\partial x}\hat{V}_2(\xi)\Big\}\Big]d\xi$$

$$\frac{\partial V_2(s)}{\partial x} = -\int_s^t \Big[\partial_x E_1(\xi,X(\xi))\frac{\partial X}{\partial x} -$$

$$c^{-1}\Big\{B(\xi,X(\xi))\partial_x\hat{V}_1(\xi) + \partial_x B(\xi,X(\xi))\frac{\partial X(\xi)}{\partial x}\hat{V}_1(\xi)\Big\}\Big]d\xi$$

Claim : $|\partial\hat{V}_i(\xi)| \leq \sqrt{2}\ \|\partial V\|$ where ∂ can be ∂_x, ∂_{v_1}, or ∂_{v_2}

Proof : Let $H(v) = \hat{v} = \dfrac{v}{\sqrt{1+c^{-2}|v|^2}}$

Then $H : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and $H_i(v) = \hat{v}_i = \dfrac{v_i}{\sqrt{1+c^2(v_1^2+v_2^2)}}$ $i = 1,2$

So $\partial\hat{V}_i = \partial(H_i(V)) = \nabla_v(H_i(V))\cdot\partial V = \nabla_v(\hat{V}_i)\cdot\partial V$

$\Rightarrow$ $|\partial\hat{V}_i| \leq \| \nabla_v\hat{V}_i \| \, \| \partial V \|$

(2.16) $\leq \sqrt{2} \, \| \partial V \|$ by lemma [2.7]

Taking absolute values, using (2.16), and the facts that $c \geq 1$ and $\dfrac{|\hat{V}_i|}{c} \leq 1$,

$i = 1,2$, we may write

$$|\partial_x X(s)| \leq 1 + \int_s^t \sqrt{2} \, \|\partial_x V(\xi)\| d\xi$$

$$|\partial_x V_1(s)| \leq \int_s^t \{(\|\partial_x E_1\|_\infty + \|\partial_x B\|_\infty)|\partial_x X(\xi)| + \sqrt{2} \, \|B\|_\infty \|\partial_x V(\xi)\|\} d\xi$$

$$|\partial_x V_2(s)| \leq \int_s^t \{(\|\partial_x E_2|_\infty + \|\partial_x B\|_\infty)|\partial_x X(\xi)| + \sqrt{2} \, \|B\|_\infty \|\partial_x V(\xi)\|\} d\xi$$

Using the bounds on the fields and their derivatives, we have

$$|\partial_x X(s)| \leq 1 + \int_s^t \sqrt{2} \, \|\partial_x V(\xi)\| d\xi$$

$$|\partial_x V_1(s)| \leq \int_s^t (D(\xi) \, |\partial_x X(\xi)| + \sqrt{2} \, D \, ||\partial_x V(\xi)||) d\xi$$

$$|\partial_x V_2(s)| \leq \int_s^t (D(\xi) \, |\partial_x X(\xi)| + \sqrt{2} \, D \, ||\partial_x V(\xi)||) d\xi$$

We add these three equations, and since $s \in [0,t]$ and $D(\xi)$ is a non-decreasing function of ξ, we have (using the triangle inequality on $||\partial V(\xi)||$)

$$|\partial_x X(s)| + |\partial_x V_1(s)| + |\partial_x V_2(s)| \leq 1 +$$

$$\int_0^t D(t) \left(|\partial_x X(\xi)| + |\partial_x V_1(\xi)| + |\partial_x V_2(\xi)| \right) d\xi$$

By Gronwall's lemma, then,

$$|\partial_x X(s)| + |\partial_x V_1(s)| + |\partial_x V_2(s)| \leq e^{tD(t)} = D(t) \quad \text{for all } s \in [0,t].$$

Now consider the v_1 derivatives. Differentiating with respect to v_1 gives

$$\partial_{v_1} X(s) = - \int_s^t \partial_{v_1} \hat{V}_1(\xi) d\xi$$

$$\partial_{v_1} V_1(s) = 1 - \int_s^t [\partial_x E_1(\xi, X(\xi)) \partial_{v_1} X(\xi) + c^{-1}(B(\xi, X(\xi)) \partial_{v_1} \hat{V}_2(\xi) +$$

$$\hat{V}_2(\xi) \partial_x B(\xi, X(\xi)) \partial_{v_1} X(\xi))] d\xi$$

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$$\partial_{v_1} V_2(s) = - \int_s^t [\partial_x E_2(\xi, X(\xi)) \partial_{v_1} X(\xi) - c^{-1}(B(\xi, X(\xi)) \partial_{v_1} \hat{V}_1(\xi) - \hat{V}_1(\xi) \partial_x B(\xi, X(\xi)) \partial_{v_1} X(\xi))] d\xi$$

By the same arguments used above, we are led to

$$|\partial_{v_1} X(s)| + |\partial_{v_1} V_1(s)| + |\partial_{v_1} V_2(s)| \leq 1 +$$

$$\int_0^t D(t)(|\partial_{v_1} X(\xi)| + |\partial_{v_1} V_1(\xi)| + |\partial_{v_1} V_2(\xi)|) d\xi)$$

and by Gronwall,

$$|\partial_{v_1} X(s)| + |\partial_{v_1} V_1(s)| + |\partial_{v_1} V_2(s)| \leq D(t), \quad \text{for all } s \in [0,t]$$

A similar argument yields the same result for the v_2 derivatives.

(4) <u>Derivatives of the Vlasov Density f:</u>

The bounds on the derivatives of the characteristics enable us now to bound the x and v derivatives of $f(t, \cdot, \cdot)$ independently of c:

$$\partial_x f(t,x,v) = \partial_x f^0(X(0,t,x,v), V(0,t,x,v))$$

$$= f_x^0(X,V) \partial_x X(0,t,x,v) + \nabla_v f^0(X,V) \cdot \partial_x V(0,t,x,v)$$

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$$\Rightarrow \quad |\partial_x f(t,x,v)| \leq \| f^0_x \|_\infty |\partial_x X(0,t,x,v)| + \|\nabla_v f^0\|_\infty \|\partial_x V(0,t,x,v)\|$$

Using the bounds on the derivatives of the characteristics and the assumptions on the data f^0, we have

$$(2.17) \qquad \|\partial_x f(t)\|_\infty \leq D(t),$$

Similarly,

$$\nabla_v f(t,x,v) \quad = \quad \nabla_v f^0(X(0,t,x,v), V(0,t,x,v))$$

$$\Rightarrow \quad \|\nabla_v f(t,x,v)\| \leq \|\partial_x f^0\|_\infty \|\nabla_v X(0,t,x,v)\| + \|\nabla_v f^0\|_\infty \|\nabla_v V(0,t,x,v)\|$$

and so

$$(2.18) \qquad \|\nabla_v f(t)\|_\infty \leq D(t)$$

(2.17) and (2.18) and the uniform bound on $f(t,x,v)$ together give

$$\|f(t)\|_{C^1} \leq D(t) \quad \text{and the lemma is proved.}$$

The t - derivative of f is bounded by using the Vlasov equation, e.g.

$$\partial_t f(t,x,v) \quad = \quad -\hat{v}_1 f_x(t,x,v) - (E + c^{-1} B M \hat{v}) \cdot \nabla_v f(t,x,v)$$

$$\Rightarrow \quad |\partial_t f(t,x,v)| \leq |\hat{v}_1| \|f_x(t)\|_\infty + (\|E(t)\|_\infty + \|B(t)\|_\infty \|\hat{v}\|) \|\nabla_v f(t)\|_\infty$$

and since $|\hat{v}_1| \leq \|\hat{v}\| \leq D(t)$ on the support of $f(t,\cdot,\cdot)$, the bounds on f_x and $\nabla_v f$ yield

$$|\partial_t f(t)| \leq D(t) \quad \text{for all} \quad t > 0, \quad \text{with} \quad D(t) \quad \text{independent of c.}$$

<u>Remark:</u>

Bounds on the t and x derivatives of j and ρ follow immediately from the bounds on the t and x derivatives of f and the compact v - support of f, using the definition of ρ as a v - integral of f. The result is

$$\|\partial j(t)\|_\infty, \ \|\partial \rho(t)\|_\infty \ \leq \ D(t) \quad \text{for all } t > 0,$$

where ∂ can be either ∂_t or ∂_x and D(t) is independent of c.

CHAPTER III: The Modified Problem

A. Modified Field Operators

We use the integral representations of the fields B and E_2:

$$E_2(t,x) = \frac{1}{2} [E_2^0(x-ct) + E_2^0(x+ct) + B^0(x-ct) - B^0(x+ct)] -$$

$$2\pi \mathcal{E}_+^c [j_2](t,x)$$

$$B(t,x) = \frac{1}{2} [E_2^0(x-ct) - E_2^0(x+ct) + B^0(x-ct) + B^0(x+ct)] -$$

$$2\pi \mathcal{E}_-^c [j_2](t,x)$$

where

$$\mathcal{E}_\pm^c [j_2](t,x) := \int_0^t [j_2(\tau, x-c(t-\tau)) \pm j_2(\tau, x+c(t-\tau))] d\tau$$

We define a modified field operator, $\tilde{\mathcal{E}}_\pm^c$ as follows:

$$(3.1) \qquad \tilde{\mathcal{E}}_\pm^c [g](t,x) := \frac{1}{c} \int_{-\infty}^x g(t,y) dy \quad \pm \quad \frac{1}{c} \int_x^\infty g(t,y) dy$$

B. Difference Between $\mathcal{E}_\pm^c$ and $\tilde{\mathcal{E}}_\pm^c$

By lemma [2.5], there is a constant D such that the support of $f(t,\cdot,v)$ is bounded by $D(1+t)^2$. Let D_0 be any constant $\geq D$.

Lemma [3.1]:   For $c > 4D_0$ and for $|y| \leq D_0(1+t^2)$

$$\left| \left(\mathcal{E}^c_\pm [j_2] - \tilde{\mathcal{E}}^c_\pm [j_2] \right) \right|_{(t,y)} \Big| \;\leq\; \begin{cases} \dfrac{D(t)}{c^2} & \text{if } t \geq 4D_0/c \\[2em] \dfrac{D(t)}{c} & \text{if } t < 4D_0/c \end{cases}$$

Proof:
$$\left(\mathcal{E}^c_\pm [j_2] - \tilde{\mathcal{E}}^c_\pm [j_2] \right)\Big|_{(t,y)} = \int_0^t j_2(\tau, y-c(t-\tau))\,d\tau \;\pm\; \int_0^t j_2(\tau, y+c(t-\tau))\,d\tau$$

$$-\left\{ \frac{1}{c} \int_{-\infty}^{y} j_2(t,\eta)\,d\eta \;\pm\; \frac{1}{c} \int_{y}^{\infty} j_2(t,\eta)\,d\eta \right\}$$

$$= \int_0^t j_2(\tau, y-c(t-\tau))\,d\tau \;-\; \frac{1}{c} \int_{-\infty}^{y} j_2(t,\eta)\,d\eta$$

$$\pm\left\{ \int_0^t j_2(\tau, y+c(t-\tau))\,d\tau \;-\; \frac{1}{c} \int_{y}^{\infty} j_2(t,\eta)\,d\eta \right\}$$

In the first integral, make the change of variable $\eta = y-c(t-\tau)$ and in the third integral, let $\eta = y+c(t-\tau)$. Then

$$\left(\mathcal{E}^c_\pm [j_2] - \tilde{\mathcal{E}}^c_\pm [j_2] \right)\Big|_{(t,y)} = \int_{y-ct}^{y} j_2\left(t - \frac{y-\eta}{c}, \eta \right) c^{-1}\,d\eta \;-\; \frac{1}{c} \int_{-\infty}^{y} j_2(t,\eta)\,d\eta$$

$$\pm \int_{y+ct}^{y} j_2\left(t + \frac{y-\eta}{c}, \eta \right)(-c^{-1})\,d\eta \;\mp\; \frac{1}{c} \int_{y}^{\infty} j_2(t,\eta)\,d\eta$$

$$= \frac{1}{c} \int_{y-ct}^{y} \left[j_2\left(t - \frac{y-\eta}{c}, \eta\right) - j_2(t,\eta)\right] d\eta - \frac{1}{c} \int_{-\infty}^{y-ct} j_2(t,\eta) d\eta$$

$$\pm \frac{1}{c} \int_{y}^{y+ct} \left[j_2\left(t + \frac{y-\eta}{c}, \eta\right) - j_2(t,\eta)\right] d\eta \mp \frac{1}{c} \int_{y+ct}^{\infty} j_2(t,\eta) d\eta$$

By the Mean Value Theorem,

$$= \frac{1}{c} \int_{y-ct}^{y} \partial_t j_2(\tau_1,\eta) \frac{|y-\eta|}{c} d\eta \pm \frac{1}{c} \int_{y}^{y+ct} \partial_t j_2(\tau_2,\eta) \frac{|y-\eta|}{c} d\eta$$

$$- \frac{1}{c} \int_{-\infty}^{y-ct} j_2(t,\eta) d\eta \mp \frac{1}{c} \int_{y+ct}^{\infty} j_2(t,\eta) d\eta$$

and so we have

(3.2)
$$\left| (\mathcal{E}_{\pm}^{c}[j_2] - \tilde{\mathcal{E}}_{\pm}^{c}[j_2]) \big|_{(t,y)} \right| \leq \frac{1}{c^2} \int_{y-ct}^{y} |\partial_t j_2(\tau_1,\eta)| |\eta-y| d\eta +$$

$$\frac{1}{c^2} \int_{y}^{y+ct} |\partial_t j_2(\tau_2,\eta)| \, |\eta-y| d\eta +$$

$$\frac{1}{c} \int_{-\infty}^{y-ct} |j_2(t,\eta)| d\eta + \frac{1}{c} \int_{y+ct}^{\infty} |j_2(t,\eta)| d\eta$$

Consider the first two integrals on the RHS. Recall, $\partial_t j_2(t,\cdot)$ is compactly supported with support bounded by a non-decreasing function

$D(t)$. Note that $\tau_1, \tau_2 \in [0,t]$. We may replace the intervals of integration with $(-D(t), D(t))$, and the first two terms on the right in (3.2) are then

$$\leq \frac{1}{c^2} \int_{-D(t)}^{D(t)} |\partial_t \dot{j}_2(\tau_1,\eta)| |\eta - y| d\eta + \frac{1}{c^2} \int_{-D(t)}^{D(t)} |\partial_t \dot{j}_2(\tau_2,\eta)| |\eta - y| d\eta$$

By the a-priori bound on $\partial_t \dot{j}_2$, we have

$|\partial_t \dot{j}_2(t,x)| \leq D(t)$ where $D(t)$ is again non-decreasing, so we can combine these two integrals as

$$\leq \frac{1}{c^2} \int_{-D(t)}^{D(t)} D(t) |\eta - y| d\eta$$

$$\leq \frac{D(t)}{c^2} \int_{-D(t)}^{D(t)} (|\eta| + |y|) d\eta$$

$$\leq \frac{D(t)}{c^2} \int_{-D(t)}^{D(t)} (D(t) + |y|) d\eta$$

$$\leq \frac{D(t)}{c^2} (|y| + 1)$$

and since $|y| \leq D(1+t^2)$,

$$\leq \frac{D(t)}{c^2}$$

We now consider the other 2 terms on the right hand side of (3.2).

Define $\mathscr{S}(t) := \{x: \exists\, v \in \mathbf{R}^2 \text{ s.t. } f(t,x,v) \neq 0\}$. Again, by the a-priori bounds and the definition of D_0, we have

$$(-D_0(1 + t^2)\,,\, D_0(1 + t^2)) \supset \mathscr{S}(t)$$

where D and D_0 are independent of t and c.

Given $y \in \mathbf{R}$, if $(y + ct_1, \infty) \cap \mathscr{S}(t_1) = \varnothing$ for some t_1, then $(y + ct, \infty) \cap \mathscr{S}(t) = \varnothing$ for all $t > t_1$, because the transport speed of f is $|\hat{v}_1| < c$. For the same reason, if $(-\infty, y - ct_2) \cap \mathscr{S}(t_2) = \varnothing$ for some t_2, then $(-\infty, y - ct) \cap \mathscr{S}(t) = \varnothing$ for all $t > t_2$.

Take $c > 4D_0$ and consider any (y,s) with $\dfrac{4D_0}{c} < s < 1$ and $|y| \le D_0(s^2 + 1)$.

Then
$$
\begin{aligned}
|y| - sc &< |y| - 4D_0 \\
&< D_0(s^2 + 1) - 4D_0 \\
&< 2D_0 - 4D_0 \\
&= -2D_0 \\
&< -D_0(s^2 + 1)
\end{aligned}
$$

So $(-\infty, |y| - cs) \cap \mathscr{S}(s) = \varnothing$ and because of transport speed, $(-\infty, |y| - ct) \cap \mathscr{S}(t) = \varnothing$ for all $t \ge s$. Similarly, $(|y| + ct, \infty) \cap \mathscr{S}(t) = \varnothing$ for all $t \ge s$.

Hence for any (y,t) with $t > \dfrac{4D_0}{c}$ and $|y| \le D_0(t^2 + 1)$,

$$\int_{-\infty}^{y-ct} |j_2(t,\eta)|\,d\eta = \int_{y+ct}^{\infty} |j_2(t,\eta)|\,d\eta = 0$$

and $\left| (\mathcal{E}^c_\pm [j_2] - \tilde{\mathcal{E}}^c_+ [j_2]) \big|_{(y,t)} \right| \le \dfrac{D(t)}{c^2}$

If $t < \dfrac{4 D_0}{c}$, the contribution of the first two integrals on the right hand side of (3.2) is still bounded by $\dfrac{D(t)}{c^2}$. There is now a contribution from the other two integrals, however. We obtain a bound on this by including in the interval of integration the piece $[y-ct, y+ct]$ and use the bounds on $|j_2|$ and its support to get

$$\frac{1}{c} \int_{-\infty}^{y-ct} |j_2(t,\eta)| d\eta + \frac{1}{c} \int_{y+ct}^{\infty} |j_2(t,\eta)| d\eta \le \frac{1}{c} \int |j_2(t,\eta)| d\eta \le \frac{D(t)}{c}$$

Result:

$$\left| (\mathcal{E}^c_\pm [j_2] - \tilde{\mathcal{E}}^c_\pm [j_2]) \big|_{(y,t)} \right| \le \begin{cases} \dfrac{D(t)}{c^2} & \text{if } t > \dfrac{4 D_0}{c} \\[2ex] \dfrac{D(t)}{c^2} + \dfrac{D(t)}{c} \le \dfrac{D(t)}{c} & \text{if } t < \dfrac{4 D_0}{c} \end{cases}$$

and the lemma is proved.

C. The Modified Problem (RVM~)

We propose a modified problem in which the field operators $\mathcal{E}^c_\pm$ are replaced with the operators $\tilde{\mathcal{E}}^c_\pm$, and in which the Cauchy data is the same:

$$(3.3) \qquad (\text{RVM}^\sim) \begin{cases} \partial_t \tilde{f} + \hat{v}_1 \partial_x \tilde{f} + (\tilde{E} + c^{-1}\tilde{B}M\hat{v})\cdot\nabla_v \tilde{f} = 0 \\ 0 \le f^0(x,v) \in C_0^1(R^3) \\ E_2^0(x) \,\&\, B^0(x) \in C_0^1(R) \end{cases}$$

The data, f^0, E_2^0, and B^0 are considered known. We also assume a smooth, compactly supported, neutralizing background density $n(x)$, as in (RVM). Charge and current densities are given by

$$\tilde{\rho}(t,x) = \int_{R^2} \tilde{f}(t,x,v)dv - n(x)$$

$$\tilde{j}(t,x) = \int_{R^2} \hat{v}\tilde{f}(t,x,v)dv$$

The electric and (scalar) magnetic fields are given by

$$\tilde{E}_1(t,x) = 4\pi \int_{-\infty}^{x} \tilde{\rho}(t,y)dy$$

$$\tilde{E}_2(t,x) = \frac{1}{2}\left[E_2^0(x-ct) + E_2^0(x+ct) + B^0(x-ct) - B^0(x+ct)\right]$$

$$- 2\pi \tilde{\mathcal{E}}_+^c[\tilde{j}_2](t,x)$$

$$\tilde{B}(t,x) = \frac{1}{2}\left[E_2^0(x-ct) - E_2^0(x+ct) + B^0(x-ct) + B^0(x+ct)\right]$$

$$- 2\pi \tilde{\mathcal{E}}_-^c[\tilde{j}_2](t,x)$$

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Chapter IV: A-priori Bounds for (RVM~)

Assuming existence of a global, compactly supported C^1 solution $\tilde{f}$, we obtain the following a-priori estimates:

A. Density Estimates

Define $\tilde{X}(s,t,x,v)$ and $\tilde{V}(s,t,x,v)$ by

$$(4.4a) \quad \begin{cases} \dot{\tilde{X}}(s,t,x,v) = \hat{\tilde{V}}_1(s,t,x,v) \\[2mm] \tilde{X}(t,t,x,v) = x \end{cases}$$

$$(4.4b) \quad \begin{cases} \dot{\tilde{V}}(s,t,x,v) = \tilde{E}(s,\tilde{X}(s)) + c^{-1}\tilde{B}(s,\tilde{X}(s))M\hat{\tilde{V}}_1(s) \\[2mm] \tilde{V}(t,t,x,v) = v \end{cases}$$

where $\tilde{X}(s)$ and $\tilde{V}(s)$ abbreviate $\tilde{X}(s,t,x,v)$ and $\tilde{V}(s,t,x,v)$. Then

$$\frac{\partial}{\partial s} \tilde{f}(s,\tilde{X}(s),\tilde{V}(s)) = 0$$

$$\Rightarrow \quad \tilde{f}(t,\tilde{X}(t),\tilde{V}(t)) = \tilde{f}(t,x,v) = \tilde{f}(0,\tilde{X}(0),\tilde{V}(0)) = f^0(\tilde{X}(0),\tilde{V}(0))$$

and since since $f^0 \in C_0^1(\mathbf{R}^3)$,

$$\|\tilde{f}(t)\|_\infty = \|f^0\|_\infty = D$$

(Non negativity of $\tilde{f}(t,x,v)$ also follows from non negativity of f^0.)

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B. Charge Conservation

We integrate the Vlasov equation in v to get

$$\int_{\mathbf{R}^2} \partial_t \tilde{f}\, dv \;+\; \int_{\mathbf{R}^2} \hat{v}_1 \partial_x \tilde{f}\, dv \;+\; \int_{\mathbf{R}^2} \left[\left(\tilde{E} + \frac{\tilde{B} M \hat{v}}{c}\right)\cdot \nabla_v \tilde{f}\right] dv \;=\; 0$$

$$=\; \partial_t \tilde{\rho} + \partial_x \tilde{j}_1 \;+\; \int_{\mathbf{R}^2}\left[\left(\tilde{E} + \frac{\tilde{B} M \hat{v}}{c}\right)\cdot \nabla_v \tilde{f}\right] dv$$

But $\left(\tilde{E} + c^{-1}\tilde{B}M\hat{v}\right)\cdot\nabla_v\tilde{f} \;=\; \nabla_v\cdot\left[\tilde{f}\left(\tilde{E} + c^{-1}\tilde{B}M\hat{v}\right)\right]$, and since $\tilde{f}\,(t,\cdot,\cdot)$ is assumed to be compactly supported, by the divergence theorem,

$$\int_{\mathbf{R}^2}\left[\left(\tilde{E} + c^{-1}\tilde{B}M\hat{v}\right)\cdot\nabla_v\tilde{f}\right] dv \;=\; 0$$

and
$$\partial_t \tilde{\rho} + \partial_x \tilde{j}_1 = 0$$

We integrate in x to get

$$\int \partial_t \tilde{\rho}\, dx \;+\; \int \partial_x \tilde{j}_1\, dx \;=\; \int \partial_t \tilde{\rho}\, dx, \quad \text{since}$$

$$\int \partial_x \tilde{j}_1\, dx \;=\; 0 \quad \text{by compact } x \text{ support of } \tilde{j}.$$

$$\Rightarrow \qquad \partial_t\left(\int \tilde{\rho}\, dx\right) = 0 \;\Rightarrow\; \int \tilde{\rho}(t,x)\, dx \;=\; \text{const}$$

$$\Rightarrow \qquad \int \left[\int \tilde{f}(t,x,v)dv - n(x)\right]dx \;\; = \;\; const \;\; = \;\; \int \left[\int \tilde{f}(0,x,v)dv - n(x)\right]dx$$

$$\Rightarrow \qquad \int\int \tilde{f}(t,x,v)dvdx \;\; = \;\; \int\int f^0(x,v)dvdx, \qquad \text{which is global charge}$$

conservation.

C. Field Estimates

1. Uniform Bound on $\tilde{E}_1$

$$\tilde{E}_1(t,x) \;\; = \;\; 4\pi \int_{-\infty}^{x} \tilde{\rho}(t,y)dy \;\; = \;\; 4\pi \int_{-\infty}^{x} \left[\int \tilde{f}(t,y,v)dv - n(y)\right]dy$$

Using the non-negativity of f and the assumptions on the data, we have

$$|\tilde{E}_1(t,x)| \;\; \leq \;\; 4\pi \int\int \tilde{f}(t,y,v)dvdy + 4\pi \int\int |n(y)|dy$$

$$= \;\; 4\pi \int\int f^0(y,v)dvdy + 4\pi \int\int |n(y)|dy$$

$$\leq \;\; D$$

2. Bounds on $\tilde{E}_2$ and $\tilde{B}$

First note that from

$$\tilde{\mathcal{E}}_{\pm}^{c}[\tilde{j}_2]\Big|_{(t,x)} \;\; = \;\; \frac{1}{c} \int_{-\infty}^{x} \tilde{j}_2(t,y)dy \;\; \pm \;\; \frac{1}{c} \int_{x}^{\infty} \tilde{j}_2(t,y)dy$$

we get, using the assumptions on f^0 and the fact that $\dfrac{|\hat{v}_2|}{c} \leq 1$,

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$$\left| \tilde{\mathcal{E}}_\pm^c \, [j_2] \right|_{(t,x)} \Big| \;\le\; \frac{1}{c} \int |\tilde{j}_2(t,y)| \, dy$$

$$\le\; \frac{1}{c} \int\!\!\int |\hat{v}_2| \tilde{f}(t,y,v) \, dv \, dy$$

$$\le\; \int\!\!\int \tilde{f}(t,y,v) \, dv \, dy$$

$$=\; \int\!\!\int f^0(y,v) \, dv \, dy \;\le\; D$$

It follows from the integral representations of $\tilde{E}_2$ and $\tilde{B}$ and the assumptions on the data functions E_2^0 and B^0 (e.g. smooth and compactly supported) that

$$| \tilde{E}_2(t,x) | \;\le\; \|E_2^0\|_\infty + \|B^0\|_\infty + 2\pi D \quad \text{and}$$

$$| \tilde{B}(t,x) | \;\le\; \|E_2^0\|_\infty + \|B^0\|_\infty + 2\pi D, \quad \forall \, x, \, t > 0$$

$\Rightarrow \qquad \|\tilde{E}_2\|_\infty$ and $\|\tilde{B}\|_\infty$ are uniformly bounded in t and x, and the bounds are independent of c.

D. Bounds on the Support of $\tilde{f}$

(1) **v - Support:** Let $P_1(t) := \sup \left\{ |v| : \exists \, x \text{ s.t. } \tilde{f}(t,x,v) \neq 0 \right\}$

Lemma [4.1]: There exists a constant D, independent of t and c, such that $P_1(t) \le D(t+1)$

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Proof :　　Integrating the characteristic equation for $\tilde{V}$ gives

$$v = \tilde{V}(0,t,x,v) + \int_0^t \left[\tilde{E}(s,\tilde{X}(s)) + c^{-1}\tilde{B}(s)M\overset{\wedge}{\tilde{V}}(s)) \right] ds$$

and since $f^0 \in C_0^1(\mathbf{R}^3)$, there is a constant D such that

$$\tilde{f}(t,x,v) = f^0(\tilde{X}(0,t,x,v), \tilde{V}(0,t,x,v)) = 0, \text{ if } |\tilde{V}(0,t,x,v)| \geq D$$

So on the support of $\tilde{f}$,

$$|v| \leq D + \int_0^t (\|\tilde{E}\|_\infty + \|\tilde{B}\|_\infty) ds$$

$$\leq D(t+1) \quad \text{by the uniform field bounds, and the lemma is}$$

proved.

(2) <u>x-Support</u>:　Let $P_2(t) := \sup \left\{ |x| : \exists v \text{ s.t. } \tilde{f}(t,x,v) \neq 0 \right\}$

Lemma [4.2]:　There exists a constant D, independent of t, x, and c, such that $P_2(t) \leq D(t^2+1)$

Proof :　　Integrating the characteristic equation for $\tilde{X}$ and using the fact that $\left| \overset{\wedge}{\tilde{V}} \right| \leq \left| \tilde{V} \right|$ gives:

$$x = \tilde{X}(0,t,x,v) + \int_0^t \overset{\wedge}{\tilde{V}}_1(s) ds$$

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$$\Rightarrow \qquad |x| \ \leq \ D + \int_0^t |\hat{\tilde{V}}(s)|ds \quad \text{on the support of } \tilde{f}$$

$$\leq \ D + \int_0^t |\tilde{V}(s)|ds$$

$$\leq \ D + Dt(t+1)$$

$$\leq \ D(1+t^2) \quad \text{where } D \text{ is independent of } t, x, \text{ and } c.$$

E. **Estimates on Derivatives**

(1) <u>Spatial Derivative of E_1</u>

From
$$\tilde{E}_1(t,x) \ = \ \int_{-\infty}^x \tilde{\rho}(t,y)dy,$$

we get
$$\partial_x \tilde{E}_1(t,x) \ = \ 4\pi \tilde{\rho}(t,x)$$

$$= \ 4\pi \left(\int \tilde{f}(t,x,v)dv - n(x) \right)$$

$$\Rightarrow \qquad |\partial_x \tilde{E}_1(t,x)| \ \leq \ 4\pi \int \tilde{f}(t,x,v)dv) + D, \quad \text{since } n \in C_0^1(R)$$

$$= \ 4\pi \left(\int_{|v| \leq P_1(t)} \tilde{f}(t,x,v)dv \right) + D$$

$$\leq \ 4\pi D \cdot D(t+1)^2 + D$$

$$\leq \ D(t+2)^2, \quad \text{where } D \text{ is independent of } t, x, \text{ and } c.$$

(2) Spatial Derivatives of $\tilde{E}_2$ and $\tilde{B}$

Recall the integral representations of $\tilde{E}_2$ and $\tilde{B}$:

$$\tilde{E}_2(t,x) = \frac{1}{2}[\text{data terms}] - \frac{2\pi}{c} \int \tilde{j}_2(t,y)dy$$

$$\tilde{B}(t,x) = \frac{1}{2}[\text{data terms}] - \frac{2\pi}{c}\left[\int_{-\infty}^{x} \tilde{j}_2(t,y)dy - \int_{x}^{\infty} \tilde{j}_2(t,y)dy\right]$$

$\partial_x\tilde{E}_2(t,x)$ is the sum of x - derivatives of data terms, which are uniformly bounded by assumption, for example:

$$|\partial_x(B^0(x-ct))| = |\partial_x B^0(x-ct)| \leq \|\partial_x B^0\|_{\infty} \leq D, \text{ since } B^0 \in C_0^1(\mathbf{R})$$

$$\Rightarrow \qquad \|\partial_x\tilde{E}_2\|_{\infty} \leq D, \quad \text{independent of } t, x, \text{ and } c$$

For $\partial_x\tilde{B}(t,x)$, in addition to the uniformly bounded data terms, we pick up an extra term from

$$-\partial_x\left\{\frac{2\pi}{c}\left(\int_{-\infty}^{x}\tilde{j}_2(t,y)dy - \int_{x}^{\infty}\tilde{j}_2(t,y)dy\right)\right\} = -\frac{4\pi}{c}\tilde{j}_2(t,x)$$

But $\qquad \dfrac{4\pi}{c}|\tilde{j}_2(t,x)| \leq \dfrac{4\pi}{c}\int |\hat{v}_2|\tilde{f}(t,x,v)dv$

$$\leq 4\pi \int_{v\text{-spt }\tilde{f}} \tilde{f}(t,x,v)dv$$

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$$\le 4\pi D(D(t+1))^2 \quad \text{by lemma [4.1]}$$

$$\le D(t+1)^2, \quad D \text{ independent of } t, x, \text{ and } c.$$

Result: $\quad \left\|\partial_x \tilde{E}_2\right\|_\infty \le D \quad$ uniformly in t and x, and

$$\left\|\partial_x \tilde{B}(t)\right\|_\infty \le D(t+1)^2 \qquad \text{for all } t \ge 0$$

(3) <u>Derivatives of characteristics</u>

We obtain bounds on ∂X, ∂V_1 and ∂V_2 where ∂ can be ∂_x, ∂_{v_1} or ∂_{v_2}.

Integrating the characteristic equations gives

$$\tilde{X}(s,t,x,v) = x - \int_s^t \hat{\tilde{V}}_1(\xi,t,x,v)d\xi$$

$$\tilde{V}_1(s,t,x,v) = v_1 - \int_s^t \left[\tilde{E}_1(\xi,\tilde{X},(\xi)) + c^{-1}\tilde{B}(\xi,\tilde{X}(\xi))\hat{\tilde{V}}_2(\xi)\right]d\xi$$

$$\tilde{V}_2(s,t,x,v) = v_2 - \int_s^t \left[\tilde{E}_2(\xi,\tilde{X}(\xi)) - c^{-1}\tilde{B}(\xi,\tilde{X}(\xi))\hat{\tilde{V}}_1(\xi)\right]d\xi$$

where $\tilde{X}(\xi)$ abbreviates $\tilde{X}(\xi,t,x,v)$, etc. We first treat the spatial derivatives:

$$\partial_x \tilde{X}(s) = 1 - \int_s^t \partial_x \hat{\tilde{V}}_1(\xi)d\xi$$

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$$\partial_x \tilde{V}_1(s) \;=\; -\int_s^t \left\{ \partial_x \tilde{E}_1(\xi,\tilde{X}(\xi))\partial_x \tilde{X}(\xi)) + c^{-1}\big[\tilde{B}(\xi,\tilde{X}(\xi))\partial_x \hat{\tilde{V}}_2(\xi) \;+ \right.$$

$$\left. \partial_x \tilde{B}(\xi,\tilde{X}(\xi))\partial_x \tilde{X}(\xi)\hat{\tilde{V}}_2(\xi)\big]\right\}d\xi$$

$$\partial_x \tilde{V}_2(s) \;=\; -\int_s^t \left\{ \partial_x \tilde{E}_2(\xi,\tilde{X}(\xi))\partial_x \tilde{X}(\xi) - c^{-1}\big[\tilde{B}(\xi,\tilde{X}(\xi))\partial_x \hat{\tilde{V}}_1(\xi) \;+ \right.$$

$$\left. \partial_x \tilde{B}(\xi,\tilde{X}(\xi))\partial_x \tilde{X}(\xi)\hat{\tilde{V}}_1(\xi)\big]\right\}d\xi$$

Take absolute values and use $\dfrac{|\hat{\tilde{V}}_1|}{c}$, $\dfrac{|\hat{\tilde{V}}_2|}{c} \leq 1$ (Also note that $s \in [0,t]$):

$$|\partial_x \tilde{X}(s)| \;\leq\; 1 + \int_s^t |\partial_x \hat{\tilde{V}}_1(\xi)|\,d\xi$$

$$|\partial_x \tilde{V}_1(s)| \;\leq\; \int_s^t \left\{ \big(|\partial_x \tilde{E}_1| + |\partial_x \tilde{B}|\big)|\partial_x \tilde{X}(\xi)| + c^{-1}|\tilde{B}||\partial_x \hat{\tilde{V}}_2(\xi)|\right\}d\xi$$

$$|\partial_x \tilde{V}_2(s)| \;\leq\; \int_s^t \left\{ \big(|\partial_x \tilde{E}_2| + |\partial_x \tilde{B}|\big)|\partial_x \tilde{X}(\xi)| + c^{-1}|\tilde{B}||\partial_x \hat{\tilde{V}}_1(\xi)|\right\}d\xi$$

Using lemma [2.7] and the a-priori bounds on the fields and their derivatives, we may write

$$|\partial_x \tilde{X}(s)| \;\leq\; 1 + \int_s^t \sqrt{2}\,\|\partial_x \tilde{V}(\xi)\|\,d\xi$$

$$|\partial_x \tilde{V}_1(s)| \leq \int_s^t \left(D(\xi)|\partial_x \tilde{X}(\xi)| + c^{-1}D\sqrt{2}||\partial_x \tilde{V}(\xi)||\right)d\xi$$

$$|\partial_x \tilde{V}_2(s)| \leq \int_s^t \left(D(\xi)|\partial_x \tilde{X}(\xi)| + c^{-1}D\sqrt{2}||\partial_x \tilde{V}(\xi)||\right)d\xi$$

Since we assume $c \geq 1$, we may disregard the c^{-1} term. Also, since $D(\xi)$ is an increasing function of ξ, $D(\xi) \leq D(t) \quad \forall \, \xi \in [s,t]$, and we have

$$|\partial_x \tilde{X}(s)| + |\partial_x \tilde{V}_1(s)| + |\partial_x \tilde{V}_2(s)| \leq 1 + \int_s^t D(t)(|\partial_x \tilde{X}(\xi)| + ||\partial_x \tilde{V}(\xi)||)d\xi$$

$$\leq 1 + \int_s^t D(t)(|\partial_x \tilde{X}(\xi)| + |\partial_x \tilde{V}_1(\xi)| + |\partial_x \tilde{V}(\xi)|)d\xi$$

Gronwall's lemma then yields

$$|\partial_x \tilde{X}(s)| + |\partial_x \tilde{V}_1(s)| + |\partial_x \tilde{V}_2(s)| \leq e^{D(t)|t-s|} \leq e^{D(t)t} = D(t) \quad \forall \, s \in [0,t]$$

and $\partial_x \tilde{X}$, $\partial_x \tilde{V}_1$, and $\partial_x \tilde{V}_2$ are all bounded independently of c. Now consider the v_1 derivatives. Differentiating the characteristic equations gives

$$\partial_{v_1} \tilde{X}(s) = - \int_s^t \partial_{v_1} \hat{\tilde{V}}_1(\xi,t,x,v)d\xi$$

and so by lemma [2.7], we have

(i) $\qquad |\partial_{v_1}\tilde{X}(s)| \leq \int_s^t \sqrt{2}\,||\partial_{v_1}\tilde{V}(\xi)||\,d\xi$

Then, $\qquad \partial_{v_1}\tilde{V}_1(s) = 1 - \int_s^t \Big\{ \partial_x\tilde{E}_1(\xi,\tilde{X}(\xi))\partial_{v_1}\tilde{X}(\xi) + c^{-1}\big[\tilde{B}(\xi,\tilde{X}(\xi))\partial_{v_1}\overset{\wedge}{\tilde{V}}_2(\xi)$

$$+ \overset{\wedge}{\tilde{V}}_2(\xi)\partial_x\tilde{B}(\xi,\tilde{X}(\xi))\partial_{v_1}\tilde{X}(\xi)\big]\Big\}\,d\xi$$

So $\qquad |\partial_{v_1}\tilde{V}_1(s)| \leq 1 + \int_s^t \Big\{ |\partial_x\tilde{E}_1(\xi)|_\infty|\partial_{v_1}\tilde{X}(\xi)| + |\tilde{B}(\xi)|_\infty|\partial_{v_1}\tilde{V}_2(\xi)|\big)$

$$+ |\partial_x\tilde{B}(\xi)|_\infty|\partial_{v_1}\tilde{X}(\xi)|\Big\}\,d\xi$$

(using $c \geq 1$ and $\dfrac{|\overset{\wedge}{\tilde{V}}_2|}{c} \leq 1$)

Now, using the a-priori bounds on $\tilde{B}$, $\partial_x\tilde{B}$, and $\partial_x\tilde{E}_1$, lemma [2.7], and the fact that $D(\xi)$ is a non-decreasing function of ξ, we have

(ii) $\qquad |\partial_{v_1}\tilde{V}_1(s)| \leq 1 + \int_s^t D(t)(|\partial_{v_1}\tilde{X}(\xi)| + ||\partial_{v_1}\tilde{V}(\xi)||)\,d\xi$

Differentiating the equation for $\tilde{V}_2$ gives

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$$\partial_{v_1}\tilde{V}_2(s) = -\int_s^t \left\{ \partial_x \tilde{E}_2(\xi,\tilde{X}(\xi))\partial_{v_1}\tilde{X}(\xi) - c^{-1}\left[\tilde{B}(\xi,\tilde{X}(\xi))\partial_{v_1}\hat{\tilde{V}}_1(\xi) \right. \right.$$

$$\left. \left. + \hat{\tilde{V}}_1(\xi)\partial_x\tilde{B}(\xi,\tilde{X}(\xi))\partial_{v_1}\tilde{X}(\xi)\right]\right\}d\xi$$

and as before we may write

(iii) $\qquad |\partial_{v_1}\tilde{V}_2(s)| \leq \int_s^t D(t)(|\partial_{v_1}\tilde{X}(\xi)| + ||\partial_{v_1}\tilde{V}(\xi)||)d\xi$

Adding i, ii, and iii gives

$$|\partial_{v_1}\tilde{X}(s)| + |\partial_{v_1}\tilde{V}_1(s)| + |\partial_{v_1}\tilde{V}_2(s)| \leq 1 + \int_s^t D(t)(|\partial_{v_1}\tilde{X}(\xi)| + ||\partial_{v_1}V(\xi)||)d\xi$$

$$\leq 1 + \int_s^t D(t)(|\partial_{v_1}\tilde{X}(\xi)| + |\partial_{v_1}\tilde{V}_1(\xi)| + |\partial_{v_1}\tilde{V}_2(\xi)|)d\xi$$

By Gronwall,

$$|\partial_{v_1}\tilde{X}(s)| + |\partial_{v_1}\tilde{V}_1(s)| + |\partial_{v_1}\tilde{V}_2(s)| \leq e^{D(t)|t-s|} \leq e^{D(t)t} = D(t) \quad \forall \ s \in [0,t].$$

A similar argument yields the same result for the v_2 derivatives.

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Chapter V. Comparison of Solutions

We have the original problem, (RVM):

$$\partial_t f + \hat{v}_1 \partial_x f + (E + c^{-1} B M \hat{v}) \cdot \nabla_v f = 0$$

with given Cauchy data

$$f(0,x,v) = f^0(x,v), \quad B^0(x), \quad \text{and} \quad E_2^0(x)$$

all smooth and of compact support. The characteristics are given by

$$\begin{cases} \dot{X}(s,t,x,v) = \hat{V}_1(s,t,x,v) \\[2mm] X(t,t,x,v) = x \end{cases}$$

$$\begin{cases} \dot{V}(s,t,x,v) = E(s,X(s)) + c^{-1} B(s,X(s)) M \hat{V}(s) \\[2mm] V(t,t,x,v) = v \end{cases}$$

The modified problem, (RVM~) is:

$$\partial_t \tilde{f} + \hat{v}_1 \partial_x \tilde{f} + (\tilde{E} + c^{-1} \tilde{B} M \hat{v}) \cdot \nabla_v \tilde{f} = 0$$

with the same Cauchy data

$$f^0(x,v), \quad B^0(x), \quad \text{and} \quad E_2^0(x)$$

The characteristics for the modified problem are given by

$$\begin{cases} \dot{\tilde{X}}(s,t,x,v) = \hat{\tilde{V}}_1 \\[2mm] \tilde{X}(t,t,x,v) = x \end{cases}$$

$$\begin{cases} \dot{\tilde{V}}(s,t,x,v) = \tilde{E}(s,\tilde{X}(s)) + c^{-1}\tilde{B}(s,\tilde{X}(s))M\hat{\tilde{V}}(s) \\[2mm] \tilde{V}(t,t,x,v) = v \end{cases}$$

The solutions of the two problems at a point (t,x,v) are

$$f(t,x,v) = f^0(X(0,t,x,v),V(0,t,x,v)) \quad \text{and}$$

$$\tilde{f}(t,x,v) = f^0(\tilde{X}(0,t,x,v),\tilde{V}(0,t,x,v))$$

Recall that by the a-priori bounds, there are c - independent constants D and $\tilde{D}$ such that the x-supports of f and $\tilde{f}$ are bounded by $D(1 + t^2)$ and $\tilde{D}(1 + t^2)$, respectively. Let $D_0 = \max(D, \tilde{D})$.

Theorem [5.1]:   There exists a non-decreasing function
$$D : [\,0, \infty) \to [\,0, \infty) \text{ such that for } c > 4D_0 \text{ and for all } t > 0,$$

$$\| f(t) - \tilde{f}(t) \|_\infty \leq \frac{D(t)}{c^2}$$

Proof :   We consider the difference of these solutions along the characteristics of the modified problem. (Note that if $|\tilde{X}(s)| \geq D_0(1 + s^2)$,

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then $\dfrac{d}{ds} (f - \tilde{f})\big|_{(s, \tilde{X}(s), \tilde{V}(s))} = 0$, so we take $|\tilde{X}(s)| < D_0(1 + s^2)$.)

$$\frac{d}{ds} (f - \tilde{f})\big|_{(s, \tilde{X}(s), \tilde{V}(s))} = \frac{d}{ds} f(s, \tilde{X}(s), \tilde{V}(s)) - 0$$

$$= \partial_t f(s, \tilde{X}(s), \tilde{V}(s)) + \partial_x f(s, \tilde{X}(s), \tilde{V}(s))\, \dot{\tilde{X}}(s) + \nabla_v f(s, \tilde{X}(s), \tilde{V}(s)) \cdot \dot{\tilde{V}}(s)$$

$$= \partial_t f(s, \tilde{X}(s), \tilde{V}(s)) + \hat{\tilde{V}}_1(s) \partial_x f(s, \tilde{X}(s), \tilde{V}(s)) +$$

$$\left[\tilde{E}(s, \tilde{X}(s)) + c^{-1} \tilde{B}(s, \tilde{X}(s)) M \hat{\tilde{V}}(s) \right] \cdot \nabla_v f(s, \tilde{X}(s), \tilde{V}(s))$$

$$= - \left\{ \hat{\tilde{V}}_1(s) \partial_x f(s, \tilde{X}(s), \tilde{V}(s)) + \left[E(s, \tilde{X}(s)) + c^{-1} B(s, \tilde{X}(s)) M \hat{\tilde{V}}(s) \right] \cdot \right.$$

$$\left. \nabla_v f(s, \tilde{X}(s), \tilde{V}(s)) \right\} + \hat{\tilde{V}}_1(s) \partial_x f(s, \tilde{X}(s), \tilde{V}(s)) +$$

$$\left[\tilde{E}(s, \tilde{X}(s)) + c^{-1} \tilde{B}(s, \tilde{X}(s)) M \hat{\tilde{V}}(s) \right] \cdot \nabla_v f(s, \tilde{X}(s), \tilde{V}(s))$$

$$= \left\{ \left[\tilde{E}(s, \tilde{X}(s)) - E(s, \tilde{X}(s)) \right] + c^{-1} \left(\tilde{B}(s, \tilde{X}(s)) M \hat{\tilde{V}}(s) - \right.\right.$$

$$\left.\left. B(s, \tilde{X}(s)) M \hat{\tilde{V}}(s) \right) \right\} \cdot \nabla_v f(s, \tilde{X}(s), \tilde{V}(s))$$

Recall the a-priori bound $\|\nabla_v f(t, \cdot, \cdot)\|_\infty \leq D(t) \; \forall\, t \geq 0$ where $D(t)$ is non decreasing. Integrating from 0 to t,

$$(f-\tilde{f})\big|_{(t,x,v)} = (f-\tilde{f})\big|_{(0,\tilde{X}(0),\tilde{V}(0))} +$$

$$\int_0^t \Big\{ \Big[\tilde{E}(s,\tilde{X}(s)) - E(s,\tilde{X}(s)) + c^{-1}\big(\tilde{B}(s,\tilde{X}(s))M\hat{\tilde{V}}(s) -$$

$$B(s,\tilde{X}(s))M\hat{\tilde{V}}(s)\big)\Big] \cdot \nabla_v f(s,\tilde{X}(s),\tilde{V}(s)) \Big\} ds$$

$$\Rightarrow \qquad \big\| (f-\tilde{f})\big|_t \big\|_\infty \leq \int_0^t \|\nabla_v f(s)\|_\infty \big\{ \|\tilde{E}(s,\tilde{X}(s)) - E(s,\tilde{X}(s))\| +$$

$$c^{-1}\|\tilde{B}(s,\tilde{X}(s))M\hat{\tilde{V}}(s) - B(s,\tilde{X}(s))M\hat{\tilde{V}}(s)\|\big\} ds$$

$$\leq \int_0^t D(t) \big\{ \|\tilde{E}(s,\tilde{X}(s)) - E(s,\tilde{X}(s))\| +$$

$$c^{-1}\|\tilde{B}(s,\tilde{X}(s))M\hat{\tilde{V}}(s) - B(s,\tilde{X}(s))M\hat{\tilde{V}}(s)\|\big\} ds$$

First consider $\| \tilde{E}(s,\tilde{X}(s)) - E(s,\tilde{X}(s)) \|$:

$$\|\tilde{E}(s,\tilde{X}(s)) - E(s,\tilde{X}(s))\| \leq |\tilde{E}_1(s,\tilde{X}(s)) - E_1(s,\tilde{X}(s))| +$$

$$|\tilde{E}_2(s,\tilde{X}(s)) - E_2(s,\tilde{X}(s))|$$

Now $\qquad |\tilde{E}_1(s,\tilde{X}(s)) - E_1(s,\tilde{X}(s))| = \left| \, 4\pi \int_{-\infty}^{\tilde{X}(s)} (\tilde{\rho}(s,y) - \rho(s,y))dy \, \right|$

$$\le \, 4\pi \int_{-\infty}^{\tilde{X}(s)} |\tilde{\rho}(s,y) - \rho(s,y)|dy$$

$$\le \, 4\pi \int_{-\infty}^{\tilde{X}(s)} \int_{\mathbf{R}^2} |\tilde{f}(s,y,v) - f(s,y,v)|dvdy$$

$$\le \, 4\pi \int\int |\tilde{f}(s,y,v) - f(s,y,v)|dvdy$$

$$\le \, D(t) \, ||\tilde{f}(s) - f(s)||_\infty \quad \text{since} \quad f(s,\cdot,\cdot) \quad \text{and}$$

$\tilde{f}(s,\cdot,\cdot)$ are compactly supported with support bounded by a non-decreasing function $D(s)$

Next, $\qquad \left| \tilde{E}_2(s,\tilde{X}(s)) - E_2(s,\tilde{X}(s)) \right| \, = 2\pi \left| \left(\tilde{\mathcal{E}}^c_+[\tilde{j}_2] - \mathcal{E}^c_+[j_2] \right) \big|_{(s,\tilde{X}(s))} \right|$

Triangulating, we write

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_+[\tilde{j}_2] - \mathcal{E}^c_+[j_2] \right) \big|_{(s,\tilde{X})} \right| \le \, 2\pi \left\{ \, \left| \left(\tilde{\mathcal{E}}^c_+[\tilde{j}_2] - \tilde{\mathcal{E}}^c_+[j_2] \right) \big|_{(s,\tilde{X})} \right| \right.$$

$$\left. + \left| \left(\tilde{\mathcal{E}}^c_+[j_2] - \mathcal{E}^c_+[j_2] \right) \big|_{(s,\tilde{X})} \right| \, \right\}$$

By lemma [4.1], we have

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_+[j_2] - \mathcal{E}^c_+[j_2] \right) \big|_{(s,\tilde{X})} \right| \leq \begin{cases} \dfrac{D(s)}{c^2} & \text{if} \quad s \geq \dfrac{4}{c} \dfrac{D_0}{c} \\[3mm] \dfrac{D(s)}{c} & \text{if} \quad s < \dfrac{4}{c} \dfrac{D_0}{c} \end{cases}$$

For the other term, we have

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_+[\tilde{j}_2] - \tilde{\mathcal{E}}^c_+[j_2] \right) \big|_{(s,\tilde{X})} \right| \leq \frac{2\pi}{c} \int \left| \tilde{j}_2(s,y) - j_2(s,y) \right| dy$$

$$\leq \frac{2\pi}{c} \int\int |\hat{v}_2| \left| \tilde{f}(s,y,v) - f(s,y,v) \right| dv\, dy$$

$$\leq 2\pi \int\int \left| \tilde{f}(s,y,v) - f(s,y,v) \right| dv\, dy$$

$$\leq 2\pi\, D(t) \left\| \tilde{f}(s) - f(s) \right\|_\infty$$

where $D(t)$ is cubic in t. So we have the following so far:

$$(5.1) \qquad \left\| f(t) - \tilde{f}(t) \right\|_\infty \leq D(t) \int_0^t \left[D(t) \left\| f(s) - \tilde{f}(s) \right\|_\infty + \begin{cases} \dfrac{D(s)}{c^2} \\[3mm] \dfrac{D(s)}{c} \end{cases} \right. +$$

$$c^{-1} \left\| \tilde{B}(s,\tilde{X}(s)) M \tilde{\hat{v}}(s) - B(s,\tilde{X}(s)) M \hat{v}(s) \right\| \Big] ds$$

where the top member of the bracketed term applies if $s \geq \frac{4D_0}{c}$ and the bottom if $s < \frac{4D_0}{c}$. We now work on the terms involving the magnetic fields.

$$c^{-1} \left\| \tilde{B}(s,\tilde{X}(s)) M \hat{\tilde{V}}(s) - B(s,\tilde{X}(s)) M \hat{\tilde{V}}(s) \right\|$$

$$\leq c^{-1} \left\| M \hat{\tilde{V}}(s) \right\| \left| \tilde{B}(s,\tilde{X}(s)) - B(s,\tilde{X}(s)) \right|$$

Then, since M is just a rotation and $\dfrac{\left| \hat{\tilde{V}}(s) \right|}{c} \leq 1$, we have

$$= c^{-1} \left\| \hat{\tilde{V}}(s) \right\| \left| \tilde{B}(s,\tilde{X},(s)) - B(s,\tilde{X}(s)) \right|$$

$$\leq \left| \tilde{B}(s,\tilde{X}(s)) - B(s,\tilde{X}(s)) \right|$$

Using the integral representations of B and $\tilde{B}$, since the data terms cancel, we get

$$\left| \tilde{B}(s,\tilde{X}(s)) - B(s,\tilde{X}(s)) \right| = \left| \left(-2\pi \tilde{\mathcal{E}}^c_{-[\tilde{j}\,2]} + 2\pi \mathcal{E}^c_{-[j2]} \right) \big|_{(s,\tilde{X}(s))} \right|$$

$$\leq 2\pi \left| \left(\tilde{\mathcal{E}}^c_{-[\tilde{j}\,2]} - \tilde{\mathcal{E}}^c_{-[j2]} \right) \big|_{(s,\tilde{X}(s))} \right| +$$

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_{-[j2]} - \mathcal{E}^c_{-[j2]} \right) \big|_{(s,\tilde{X}(s))} \right|$$

Again by lemma [4.1] we have

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_-[j_2] - \mathcal{E}^c_-[j_2] \right) \Big|_{(s,\tilde{x}(s))} \right| \le \begin{cases} \dfrac{D(s)}{c^2} & \text{if} \quad s \ge \dfrac{4\,D_0}{c} \\[2em] \dfrac{D(s)}{c} & \text{if} \quad s < \dfrac{4\,D_0}{c} \end{cases}$$

Considering the other term, we have

$$2\pi \left| \left(\tilde{\mathcal{E}}^c_-[\tilde{j}_2] - \tilde{\mathcal{E}}^c_-[j_2] \right) \Big|_{(s,\tilde{x}(s))} \right| = 2\pi \left| \frac{1}{c} \int_{-\infty}^{\tilde{x}(s)} \tilde{j}_2(t,y)\,dy \;-\; \frac{1}{c} \int_{\tilde{x}(s)}^{\infty} \tilde{j}_2(t,y)\,dy \right.$$

$$\left. - \left(\frac{1}{c} \int_{-\infty}^{\tilde{x}(s)} j_2(t,y)\,dy \;-\; \frac{1}{c} \int_{\tilde{x}(s)}^{\infty} j_2(t,y)\,dy \right) \right|$$

$$= \frac{2\pi}{c} \left| \int_{-\infty}^{\tilde{x}(s)} \left(\tilde{j}_2(t,y) - j_2(t,y) \right) dy \;-\; \int_{\tilde{x}(s)}^{\infty} \left(\tilde{j}_2(t,y) - j_2(t,y) \right) dy \right|$$

$$\le \frac{2\pi}{c} \left\{ \int_{-\infty}^{\tilde{x}(s)} \left| \tilde{j}_2(t,y) - j_2(t,y) \right| dy + \int_{\tilde{x}(s)}^{\infty} \left| \tilde{j}_2(t,y) - j_2(t,y) \right| dy \right\}$$

$$= \frac{2\pi}{c} \int \left| \tilde{j}_2(t,y) - j_2(t,y) \right| dy$$

and by the same steps as before, this is

$$\le 2\pi D(t) \, \| \tilde{f}(s) - f(s) \|_\infty$$

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So from the $c^{-1} ||\tilde{B}M\hat{\tilde{V}} - BM\hat{\tilde{V}}||$ term, we pick up two additional terms under the integral sign on the right hand side of (5.1), namely:

$$2\pi \, D(t) \, ||\tilde{f}(s) - f(s)||_\infty + \begin{cases} \dfrac{D(s)}{c^2} \\[2mm] \dfrac{D(s)}{c} \end{cases}$$

Result:

$$(5.2) \qquad || f(t) - \tilde{f}(t) ||_\infty \leq D(t) \int_0^t || f(s) - \tilde{f}(s)||_\infty + \left\{ \begin{array}{c} \dfrac{D(s)}{c^2} \\[2mm] \dfrac{D(s)}{c} \end{array} \right\} ds$$

where the top term in brackets applies when $s \geq 4D_0/c$ and the bottom when $s < 4D_0/c$.

Case 1: $t < \dfrac{4D_0}{c}$. Then $s < 4D_0/c$ for all $s \in [0,t]$ so the bottom term in brackets in (5.2) applies, and

$$||f(t) - \tilde{f}(t)||_\infty \leq D(t) \int_0^t \left\{ || f(s) - \tilde{f}(s) ||_\infty + \dfrac{D(t)}{c} \right\} ds$$

$$\leq D(t) \left[\int_0^t || f(s) - \tilde{f}(s) ||_\infty + \int_0^{4D_0/c} \dfrac{D(t)}{c} ds \right]$$

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$$\leq \frac{D(t)}{c^2} + D(t) \int_0^t \| f(s) - \tilde{f}(s) \|_\infty \, ds$$

By Gronwall,

$$(5.3) \qquad \| f(t) - \tilde{f}(t) \|_\infty \leq \frac{D(t)}{c^2} \exp(tD(t)) = \frac{D(t)}{c^2} \quad \text{for all } t < \frac{4D_0}{c}$$

Case 2: $t \geq \frac{4D_0}{c}$. Then from $s = 0$ to $s = \frac{4D_0}{c}$, the same argument applies, and the error is bounded by $\frac{D(t)}{c^2}$.

For $s \in [4D_0/c, t]$, the $\frac{1}{c^2}$ error term applies, and we have

$$\| f(t) - \tilde{f}(t) \|_\infty \leq \frac{D(t)}{c^2} + \int_{4D_0/c}^t \left\{ D(t) \| f(s) - \tilde{f}(s) \|_\infty + \frac{D(t)}{c^2} \right\} ds$$

$$\leq \frac{D(t)}{c^2} + \int_0^t D(t) \| f(s) - \tilde{f}(s) \|_\infty ds + \frac{tD(t)}{c^2}$$

$$\leq \frac{D(t)}{c^2} + \int_0^t D(t) \| f(s) - \tilde{f}(s) \|_\infty ds$$

By Gronwall, we again have

$$\| f(t) - \tilde{f}(t) \|_\infty \leq \frac{D(t)}{c^2}$$

Result:

(5.4)
$$\| f(t) - \tilde{f}(t) \|_\infty \leq \frac{D(t)}{c^2} \quad \text{for all } t > 0$$

and the theorem is proved. An immediate result is the following

Corollary [5.1]:   Let $S(t) = \{$ x: there exists v with either $f(t,x,v) \neq 0$ or $\tilde{f}(t,x,v) \neq 0 \}$. (So $(- D_0(1 + t^2), D_0(1 + t^2)) \supset S(t)$.) Then for all $t > 4D_0/c$,

$$\| E(t) - \tilde{E}(t) \|_s + \| B(t) - \tilde{B}(t) \|_s \leq \frac{D(t)}{c^2}$$

where $\| g(t) \|_s$ denotes the supremum over all $x \in S(t)$ of $g(t,x)$.

The proof follows immediately from theorem [5.1], the definitions of E, $\tilde{E}$, B, $\tilde{B}$, and the c-independent bounds on the x and v support of f and $\tilde{f}$.

CHAPTER VI: Computing Solutions

A.   The Program

The scheme used is of the type appearing in [3], e.g. a particle method is used on the Vlasov equation while the fields are advanced by using the exact solution representation applied to approximate solutions. In [3], Glassey and Schaeffer proved that such a scheme for RVM is first order in space and time. We take $c=1$ and begin by choosing a phase space grid.

Let Δx, Δv_1, Δv_2, Δt be > 0.

Define: $C^\alpha = \{ (x,v) : \alpha_1 \Delta x \le x \le (\alpha_1+1)\Delta x,\ \alpha_2 \Delta v_1 \le v_1 \le (\alpha_2+1)\Delta v_1,\ \alpha_3 \Delta v_2 \le$

$$v_2 \le (\alpha_3+1)\Delta v_2 \}$$

and $C^\alpha = \{ (\alpha_1+1/2)\Delta x,\ (\alpha_2+1/2)\Delta v_1,\ (\alpha_3+1/2)\Delta v_2 \}$

Let $q^\alpha = f^0(C^\alpha)(\Delta x)(\Delta v_1)(\Delta v_2) \equiv \int_{C^\alpha} f^0 dv dx$

(This will be the charge of a particle whose initial state is C^α.)

Let $\mathcal{C} = \{ \alpha \in \mathbf{Z}^3 : q^\alpha \ne 0 \}$ and note that $\mathcal{C}$ is finite by hypothesis.

Let $\varepsilon = \Delta x$ and define S, δ_ε, and Θ_ε by

$$S(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$\delta_\varepsilon(x) = \varepsilon^{-1} S(\varepsilon^{-1} x)$$

$$\Theta_\varepsilon(x) = \int_{-\infty}^{x} \delta_\varepsilon(y)\, dy$$

Define a grid on space-time ($[0,\infty) \times \mathbf{R}$) as follows:

$$t^n = n\Delta t, \quad n = 0, 1/2, 1, 3/2, \ldots$$
$$x^k = k\Delta x, \quad k \in \mathbf{Z}$$

To start the simulation, define approximate quantities (designated for the moment by underlines) as follows: For all $k \in \mathbf{Z}$ and $\alpha \in \mathbf{Z}^3$,

$$\underline{\tilde{E}}(0, x^k) = E^0(x^k)$$
$$\underline{\tilde{B}}(0, x^k) = B^0(x^k)$$
$$\underline{\tilde{X}}^\alpha(0) = (\alpha_1 + 1/2)\Delta x$$
$$\underline{\tilde{V}}^\alpha(t^{1/2}) = \tilde{V}(t^{1/2}, 0, C^\alpha)$$

To define the simulation iteratively, assume that for some n, $\underline{\tilde{E}}(t^n, x^k)$, $\underline{\tilde{B}}(t^n, x^k)$, $\underline{\tilde{X}}^\alpha(t^n)$, and $\underline{\tilde{V}}^\alpha(t^{n+1/2})$ are known approximations of $\tilde{E}(t^n, x^k)$, $\tilde{B}(t^n, x^k)$, $\tilde{X}(t^n, 0, C^\alpha)$, and $\tilde{V}(t^{n+1/2}, 0, C^\alpha)$ for all $k \in \mathbf{Z}$ and $\alpha \in \mathbf{Z}^3$.

At this point, in order to simplify the notation, we drop the tildas (which designate quantities from the modified problem) and the underlines (designating approximate quantities). For the remainder of this section,

all quantities are approximate quantities for the modified problem.
First, advance X^α by defining

$$X^\alpha(t) = X^\alpha(t^n) + (t - t^n)\hat{V}^\alpha_1(t^{n+1/2}) \qquad \forall \, t \in [t^n, \ t^{n+1}].$$

The next step is to compute and advance the approximate sources ρ and j_2.

From
$$f(t^{n+1}, x, v) = \sum_\alpha q^\alpha \delta_\varepsilon(x - X^\alpha(t^{n+1})) \delta_\varepsilon(v - V^\alpha(t^{n+1/2}))$$

we get
$$\rho(t^{n+1}, x) = \sum_\alpha q^\alpha \delta_\varepsilon(x - X^\alpha(t^{n+1})) - n(x)$$

and
$$j_2(t^{n+1}, x) = \sum_\alpha q^\alpha \hat{V}^\alpha_2(t^{n+1/2}) \delta_\varepsilon(x - X^\alpha(t^{n+1}))$$

Next, the fields are advanced using these approximate sources. We know $\rho(t^{n+1}, x)$ and $j_2(t^{n+1}, x)$ for all x. To advance E_1:

$$E_1(t^{n+1}, x^k) = 4\pi \int_{-\infty}^{x^k} \rho(t^{n+1}, y)\, dy$$

$$= 4\pi \int_{-\infty}^{x^k} \left[\sum_\alpha q^\alpha \delta_\varepsilon(y - X^\alpha(t^{n+1})) - n(y) \right] dy$$

$$= 4\pi \sum_\alpha q^\alpha \int_{-\infty}^{x^k} \delta_\varepsilon(y - X^\alpha(t^{n+1}))\, dy - 4\pi \int_{-\infty}^{x^k} n(y)\, dy$$

$$= 4\pi \sum_\alpha q^\alpha \Theta_\varepsilon(x - X^\alpha(t^{n+1})) - 4\pi \int_{-\infty}^{x^k} n(y)\, dy$$

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To advance E_2 and B using the approximate source j_2, we employ the integral representations to get

$$E_2(t^{n+1}, x^k) = \frac{1}{2}\left[E_2^0(x^k-t^{n+1}) + E_2^0(x^k+t^{n+1}) + B^0(x^k-t^{n+1}) - B^0(x^k+t^{n+1})\right]$$

$$- 2\pi \int j_2(t^{n+1}, y)\, dy$$

$$B(t^{n+1}, x^k) = \frac{1}{2}\left[E_2^0(x^k-t^{n+1}) - E_2^0(x^k+t^{n+1}) + B^0(x^k-t^{n+1}) + B^0(x^k+t^{n+1})\right]$$

$$-2\pi\left[\int_{-\infty}^{x^k} j_2(t^{n+1}, y)\, dy - \int_{x^k}^{\infty} j_2(t^{n+1}, y)\, dy\right]$$

The data terms are known. To treat the integrals, we proceed as follows:

$$2\pi \int j_2(t^{n+1}, y)\, dy = 2\pi \int \left[\sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\delta_\varepsilon(y-X^\alpha(t^{n+1}))\right] dy$$

$$= 2\pi \sum_\alpha q^\alpha \int \left[\hat{V}_2^\alpha(t^{n+1/2})\delta_\varepsilon(y-X^\alpha(t^{n+1}))\right] dy$$

$$= 2\pi \sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2}) \int \delta_\varepsilon(y-X^\alpha(t^{n+1}))\, dy$$

$$= 2\pi \sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})$$

and for the integrals in the expression for B, we have

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$$\int_{-\infty}^{x^k} j_2(t^{n+1},y)dy = \int_{-\infty}^{x^k}\left[\sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\delta_\varepsilon(y-X^\alpha(t^{n+1}))\right]dy$$

$$= \left[\sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\right]\Theta_\varepsilon(x-X^\alpha(t^{n+1}))$$

and
$$\int_{x^k}^{\infty} j_2(t^{n+1},y)dy = \int_{x^k}^{\infty}\left[\sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\delta_\varepsilon(y-X^\alpha(t^{n+1}))\right]dy$$

$$= \sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\int_{x^k}^{\infty}\delta_\varepsilon(y-X^\alpha(t^{n+1}))dy$$

$$= \sum_\alpha q^\alpha \hat{V}_2^\alpha(t^{n+1/2})\left[1-\Theta_\varepsilon(x-X^\alpha(t^{n+1}))\right]$$

The result is that the fields are advanced using

$$E_1(t^{n+1},x^k) = 4\pi\sum_\alpha q^\alpha\Theta_\varepsilon(x-X^\alpha(t^{n+1})) - 4\pi\int_{-\infty}^{x^k} n(y)dy$$

$$E_2(t^{n+1},x^k) = \frac{1}{2}\left[E_2^0(x^k-t^{n+1}) + E_2^0(x^k+t^{n+1}) + B^0(x^k-t^{n+1}) - B^0(x^k+t^{n+1})\right]$$

$$- 2\pi\sum_\alpha q^\alpha\hat{V}_2^\alpha(t^{n+1/2})$$

$$B(t^{n+1},x^k) = \frac{1}{2}\left[E_2^0(x^k-t^{n+1}) - E_2^0(x^k+t^{n+1}) + B^0(x^k-t^{n+1}) + B^0(x^k+t^{n+1})\right]$$

$$+ 2\pi\sum_\alpha q^\alpha\hat{V}_2^\alpha(t^{n+1/2})\left[1 - 2\Theta_\varepsilon(x-X^\alpha(t^{n+1}))\right]$$

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We now define the fields $E(t^{n+1},x)$ and $B(t^{n+1},x)$ $\forall\, x \in \mathbf{R}$ by linear interpolation:

$$B(t^{n+1},x) = \sum_k B(t^{n+1},x^k)\varepsilon\delta_\varepsilon(x-x^k)$$

with similar expressions for E_1 and E_2. The final step is to advance the momenta. We take

$$V_1^\alpha(t^{n+3/2}) = V_1^\alpha(t^{n+1/2}) + \Delta t E_1(t^{n+1},X^\alpha(t^{n+1})) +$$

$$\frac{\Delta t}{2} B(t^{n+1},X^\alpha(t^{n+1}))\left[\hat{V}_2^\alpha(t^{n+1/2}) + \hat{V}_2^\alpha(t^{n+3/2})\right]$$

Similarly,

$$V_2^\alpha(t^{n+3/2}) = V_2^\alpha(t^{n+1/2}) + \Delta t E_2(t^{n+1},X^\alpha(t^{n+1})) -$$

$$\frac{\Delta t}{2} B(t^{n+1},X^\alpha(t^{n+1}))\left[\hat{V}_1^\alpha(t^{n+1/2}) + \hat{V}_1^\alpha(t^{n+3/2})\right]$$

Or, as a system,

$$V^\alpha(t^{n+3/2}) = V^\alpha(t^{n+1/2}) + \Delta t E(t^{n+1},X^\alpha(t^{n+1})) +$$

$$\frac{\Delta t}{2} B(t^{n+1},X^\alpha(t^{n+1}))M\left[\hat{V}_1^\alpha(t^{n+1/2}) + \hat{V}_1^\alpha(t^{n+3/2})\right]$$

This nonlinear system is always uniquely solvable for $V^\alpha(t^{n+3/2})$ and this completes one step of the scheme.

The program was tested extensively, first by comparing against single particle problems for which exact solutions could be computed, and then

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by generating steady-state solutions (see appendix A) which also provided "exact" solutions against which the program's output could be directly compared. We found the scheme to be better than first order accurate in x,v, and t, but not quite second order.

CHAPTER VII: Numerical Experiments

A. Scaling

The analysis of the convergence of solutions of (RVM) and (RVM˜) as a function of the (increasing) parameter c was necessarily done with c appearing explicitly.   We hoped to demonstrate this convergence by direct comparison of computed solutions of the particle scheme coded for the modified problem with those produced by the Glassey/Schaeffer scheme from [3], given the same Cauchy data. A complication is that in [3], the speed of light is taken to be 1 (as are the charge q and the electron rest mass m) and in order to compare the output of the two programs when provided with the same data, we re-scale the modified problem to achieve c = 1. (See appendix B) The analog of the parameter c becoming large in the unscaled problem then is the v-support of the data (and solutions) becoming small in the scaled problems.

B.   Data

After preliminary tests on simplified problems (for instance problems with $E_2^0 \equiv B^0 \equiv 0$), we analyzed results with 2 basic sets of Cauchy data, providing a "symmetric" problem and an "asymmetric" one.

1. Symmetric Problem

For the first problem, the data is as follows: Define

(7.1) $f^0(x,v_1,v_2) = [(1-x^2)(1-v_1^2)(1-v_2^2)]^3$ for $|x| < 1,$ $|v_1| < 1,$ $|v_2| < 1$

and $f^0 = 0$ otherwise.

Then, since $\int_{-1}^{1}(1-s^2)^3 ds = \frac{32}{35}$, we see that

$$\int f^0 dv = \left(\frac{32}{35}\right)^2 (1-x^2)^3 \text{ for } |x| < 1 \text{ and } 0 \text{ for } |x| \geq 1$$

Define $n(x) = \left(\frac{32}{35}\right)^2 (1-x^2)^3 + 2x(1-x^2)^5$

Then the condition of global neutrality is met, since the first term in n cancels with $\int f^0 dv$ and the second term is odd, hence has integral zero. Note also that the smoothness requirements on the data are met.

As data for E_2 and B, we choose

$$E_2^0 = a_1(1-x^2)^3$$

$$B^0 = a_2(1-x^2)^3$$

where the parameters a_1 and a_2 allow us to adjust the size of the data.

The above is for the unscaled problem, i.e., the case in which the data are fixed and c is allowed to increase. For purposes of computation and comparison, we scale to $c=1$ and allow the support of the data to decrease. As discussed in appendix B, we take, using bars to designate quantities in the $c=1$ problem,

$$\overline{f^0}(x,v) = c^2 f^0(cx,cv)$$

$$= c^2[(1-c^2x^2)(1-c^2v_1^2)(1-c^2v_2^2)]^3 \text{ for } |x|, |v_1|, |v_2| < \frac{1}{c}$$

and $\overline{f^0} = 0$ otherwise

$$\overline{E_0^2}(x) = \frac{a_1}{c}(1-c^2x^2)^3$$

$$\overline{B^0}(x) = \frac{a_2}{c}(1-c^2x^2)^3$$

Also,

$$\overline{n}(x) = n(cx)$$

$$= \left(\frac{3\,2}{3\,5}\right)^2 (1-c^2x^2)^3 + 2cx(1-c^2x^2)^5 \text{ if } |x| < \frac{1}{c} \text{ and}$$

$$\overline{n}(x) = 0 \text{ otherwise}$$

It is clear that allowing c to increase in the unscaled problem corresponds to the support of the data in the scaled problem decreasing like $\frac{1}{c}$.

2. Asymmetric Problem

In the second problem, the data functions above are shifted as follows:

We translate f^0 as given in (7.1) by 1 in all three coordinate directions in phase space.

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$$f^0(x,v) = \left[(1-(x-1)^2)(1-(v_1-1)^2)(1-(v_2-1)^2)\right]^3$$

$$\text{if } 0 \leq x, v_1, v_2 \leq 2 \text{ and}$$

$$f^0 = 0 \quad \text{otherwise}$$

Then $\int_0^2 \left[1-(s-1)^2\right]^3 ds = \frac{32}{35}$ and we define

$$n(x) = \left(\frac{32}{35}\right)^2 (1-(x-1)^2)^3 + 2(x-1)(1-(x-1)^2)^5$$

and, as in the symmetric problem, we have global neutrality. In a similar way, we set

$$E_0^2 = a_1(1-(x-1)^2)^3$$

$$B^0 = a_2(1-(x-1)^2)^3$$

In this case, the initial conditions provided for the scaled problems are

$$\overline{f}^0(x,v) = c^2\left[(1-(cx-1)^2)(1-(cv_1-1)^2(1-(cv_2-1)^2)\right]^3$$

$$\text{if } 0 \leq x, v_1, v_2 \leq \frac{2}{c} \text{ and } 0 \text{ otherwise.}$$

$$\overline{E}_0^2(x) = \frac{a_1}{c}\left[1-(cx-1)^2\right]^3$$

$$\overline{B}^0 = \frac{a_2}{c}\left[1-(cx-1)^2\right]^3$$

Also,

$$\overline{n}(x) = \left(\frac{32}{35}\right)^2 \left[1-(cx-1)^2\right]^3 + 2(cx-1)\left[1-(cx-1)^2\right]^5$$

$$\text{if } 0 \leq x \leq \frac{2}{c}$$

and $\overline{n}(x) = 0$ otherwise.

Chapter VIII: Summary of Results

A.   Expected convergence rates

In attempting to observe the convergence rate of the solutions as c increases, we compared the values of the fields E_2 and $\tilde{E}_2$ on the union of the support of f and the support of $\tilde{f}$, i.e. where there is charge. (Recall that $\tilde{E}_2(t,\cdot)$ is not compactly supported.)

We chose values of c of 25, 50, 100, and 200, and compared the outputs of the two programs at $t = 0.08$. Since the scheme from [3] operates "at the CFL boundary", e.g. $\Delta x = \Delta t$, a balance had to be struck between a small enough Δx (which determines the number of particles) for good resolution, and a reasonable number of timesteps to avoid excessive run times. We settled on total particle numbers of 64,000 for the modified program and 59,319 for the unmodified one, (The discrepancy is due to a slight difference in the way the programs initialize the particles) with corresponding Δx's ranging from 0.002 for the $c=25$ runs to 0.00025 for $c=200$. With $\Delta t = \Delta x$, this resulted in 10 timesteps for $c=25$, increasing to 320 timesteps for $c=200$.

We were interested in documenting the following 3 main results:

1. Convergence of solutions of the two problems as c grows. According to corollary [5.1], we should observe

$$\| E_2(0.08) - \tilde{E}_2(0.08) \|_s \le \frac{D}{c^2}$$

This result applies to the unscaled problem. For the scaled problems, we have

$$\overline{E}_2(t,x) = c^{-1} E_2(t,cx)$$

$$\overline{\tilde{E}}_2(t,x) = c^{-1}\tilde{E}_2(t,cx)$$

so $\| E_2(t) - \tilde{E}_2(t) \|_s = \| c\overline{E}_2(t) - c\overline{\tilde{E}}_2(t) \|_s$

$\Rightarrow \qquad \| \overline{E}_2(t) - \overline{\tilde{E}}_2(t) \|_s \leq \dfrac{D(t)}{c^3}$

2. Avoidance of the CFL restriction in the modified problem,
    e.g. attainment of comparable agreement with the
    program for the unmodified problem when the modified
    program uses much larger timesteps.

3. Improvements in accuracy in solutions of the modified
    problem over those computed without the data terms
    present in the solution representation. A significant
    difference here would demonstrate the value of the
    appearance of the data terms involving the original Cauchy
    data, which again would not be present were the Darwin
    and quasi-electrostatic modifications of Maxwell's
    equations made first.

B. Results

We first give results for the symmetric problem.

c	Δt	max $\lvert E_2 - \tilde{E}_2 \rvert$ with $\tilde{\Delta t} = \Delta t$		max $\lvert E_2 - \tilde{E}_2 \rvert$ with large $\tilde{\Delta t}$	max $\lvert E_2 - \tilde{E}_2 \rvert$ with no data terms		$\dfrac{col6}{col3}$
25	0.002	3.514E-03		3.532E-03	1.347E-02		3.8
50	0.001	7.120E-05	49.3	7.295E-05	1.680E-03	8.0	23.6
100	0.0005	9.057E-06	7.9	9.471E-06	2.102E-04	8.0	23.2
200	0.00025	1.135E-06	8.0	$\tilde{\Delta t}$ = 0.004: 1.180E-06 $\tilde{\Delta t}$ = 0.008: 9.546E-06	2.629E-05	8.0	23.2

Remarks : 1. Convergence rate - Column 4 shows that as c is doubled, the solutions converge at the rate of $1/c^2$, as predicted in Chapter V.

2. Avoidance of CFL limitation - The value of $\tilde{\Delta t}$ used in the c=25,50,100 runs was 0.008, which in the modified scheme yielded the same accuracy as the CFL-limited unmodified scheme. For the c=100 runs, this gives a factor of 16 in the size of the timestep. In the c=200 run, we achieved the same accuracy with $\tilde{\Delta t}$ =0.004, which is again a factor of 16 better than the unmodified scheme.

3. Inclusion of data terms - Column 8 shows a factor of approximately 23 in the accuracy of the solutions when the data terms are present in the solution representation vs

when they are omitted.

The analagous results for the asymmetric problem are

c	Δt	max $\|E_2-\tilde{E}_2\|$ with $\tilde{\Delta t}=\Delta t$		max $\|E_2-\tilde{E}_2\|$ with large $\tilde{\Delta t}$	max $\|E_2-\tilde{E}_2\|$ with no data terms		$\dfrac{\text{col6}}{\text{col3}}$
25	0.002	4.603E-03		4.630E-03	1.443E-02		3.1
50	0.001	1.337E-04	33.4	1.408E-04	1.762E-03	8.2	12.8
100	0.0005	1.588E-05	8.7	1.650E-05	2.177E-04	8.1	13.7
200	0.00025	1.916E-06	8.3	$\tilde{\Delta t}=0.004$: 1.972E-06 $\tilde{\Delta t}=0.008$: 1.016E-05	2.706E-05	8.0	14.1

Remarks : 1. The $1/c^2$ convergence rate is again shown by the entries in column 4.

2. Allowable $\tilde{\Delta t}$ is again 16 times as large as in the unmodified problem.

3. Although not as great as for the symmetric problem, the increase in accuracy resulting from the inclusion of the data terms is still better than an order of magnitude.

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Appendix A: Steady State Solutions for   (RVM)

We seek a solution of the time - independent Vlasov equation   (with $c = 1$)

(A.1) $\hat{v}_1 \partial_x f(x,v) + (E(x) + c^{-1}B(x)M\hat{v}) \cdot \nabla_v f(x,v) = 0$

We impose the condition $E_2 \equiv 0$. Since in the time - independent (and 1 space, 2 momenta) case, Faraday's Law of Induction becomes

$\partial_x E_2 = 0$

$\Rightarrow$ $E_2 \equiv$ const, and $E_2 \equiv 0$ is the only finite energy solution. The equations of the characteristics are then

$\dot{X} = \hat{V}_1$

$\dot{V}_1 = E_1(X) + \hat{V}_2 B(X)$

$\dot{V}_2 = -\hat{V}_1 B(X)$

Since E_1 is an electrostatic field, there is a potential, $\mathcal{U}$, such that

$E_1(x) = -\mathcal{U}'(x)$

We introduce $\mathcal{B}$, an anti - derivative of B,   e.g.

$B(x) = \mathcal{B}'(x)$

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Consider the quantities

$$\mathcal{E} := \sqrt{1+|v|^2} + \mathcal{U}$$

$$\ell := v_2 + \mathcal{B}$$

Along characteristics of the time - independent Vlasov equation, these quantities are conserved:

$$\frac{d}{dt}\mathcal{E}(X(t),V(t)) = \partial_x\mathcal{E}\,\dot{X} + \nabla_v\mathcal{E}\cdot\dot{V}$$

$$= \mathcal{U}'(X)\dot{X} + \frac{V\cdot\dot{V}}{\sqrt{1+|V|^2}}$$

$$= \hat{V}_1\cdot\dot{V}_1 + \hat{V}_2\cdot\dot{V}_2 + \mathcal{U}'(X)\dot{X}$$

$$= \hat{V}_1[\,-\mathcal{U}'(X) + \hat{V}_2\mathcal{B}'(X)] + \hat{V}_2(-\hat{V}_1\mathcal{B}'(X)) + \mathcal{U}'(X)\hat{V}_1$$

$$= 0$$

$$\frac{d}{dt}\big(V_2(t) + \mathcal{B}(X(t))\big) = \dot{V}_2 + \mathcal{B}'(X)\dot{X}$$

$$= -\hat{V}_1\mathcal{B}'(X) + \mathcal{B}'(X)\hat{V}_1$$

$$= 0$$

It follows that $f(x,v) := g(\mathcal{E},\ell)$ is a solution of (1) for any (sufficiently

smooth) function g.

The potential $\mathcal{U}$ satisfies Poisson's equation:

$$-\mathcal{U}'' = 4\pi\rho = 4\pi\left(\int f dv - n \right) \quad \text{or}$$

(A.2) $$-\mathcal{U}''(x) = 4\pi\left[\int g\left(\sqrt{1+|v|^2} + \mathcal{U}(x), v_2 + \mathcal{B}(x)\right)dv - n(x)\right]$$

In the time - independent case, Ampere's Law becomes

$$-\partial_x B = 4\pi j_2, \quad \text{so} \quad \mathcal{B} \quad \text{satisfies}$$

$$-\mathcal{B}'' = 4\pi j_2 = 4\pi \int \hat{v}_2 f dv, \quad \text{or}$$

(A.3) $$-\mathcal{B}''(x) = 4\pi \int \hat{v}_2 g\left(\sqrt{1+|v|^2} + \mathcal{U}(x), v_2 + \mathcal{B}(x)\right)dv$$

We have then equations satisfied by $\mathcal{U}$ and $\mathcal{B}$ and the task is to find $\mathcal{U}$, $\mathcal{B}$, g, and n so that (A.2) and (A.3) are satisfied, neutrality holds, and the appropriate requirements for compact support and (to the extent possible) smoothness are met.

For simplification, we will require (A.2) and (A.3) to hold on (0,1). The various functions appearing will be defined on (-1,0) as even or odd extensions, and outside (-1,1) as constants. With these requirements in mind, we impose

(BC) $$\mathcal{U}'(0) = \mathcal{U}'(1) = \mathcal{B}'(0) = \mathcal{B}'(1) = 0$$

$\mathcal{U}$ and $\mathcal{B}$ are extended evenly on $(-1,0)$ and as constants on $\mathbf{R}\backslash(-1,1)$ as follows:

$$\mathcal{U}(x) = \mathcal{U}(1) = 0, \quad |x| \geq 1$$
$$\mathcal{U}(-x) = \mathcal{U}(x)$$

$$\mathcal{B}(x) = \mathcal{B}(1), \quad |x| \geq 1$$
$$\mathcal{B}(-x) = \mathcal{B}(x)$$

For n, we require

$$n(x) = 0, \quad |x| \geq 1$$
$$n(-x) = n(x)$$

On g, we impose the condition

$$g(\mathcal{E}, \ell) = 0 \quad \text{if } \mathcal{E} \geq 1$$

Note that this ensures the compact x-support of f, since

$$|x| \geq 1 \implies \mathcal{U}(x) = 0$$

$$\implies \mathcal{E} = \sqrt{1+|v|^2} + \mathcal{U} \geq 1$$

$$\implies g(\mathcal{E}, \ell) =: f(x,v) = 0$$

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With these definitions, (A.2) and (A.3) are satisfied $\forall$ x. Note also that the neutrality condition is built into the boundary conditions, since

$$\int \rho \, dx = 1/4\pi \int E_1' dx = \frac{1}{4\pi} (E_1(1) - E_1(-1))$$

$$= \frac{1}{4\pi} \left(\mathcal{U}'(-1) - \mathcal{U}'(1) \right) = 0$$

We now proceed to choose g. Let

$$g(\mathcal{E}, \ell) = \ell^2 I_{(-\infty, 1)}(\mathcal{E})$$

$$= \left(v_2^2 + 2v_2\mathcal{B} + \mathcal{B}^2 \right) I_{(-\infty, 1)}(\mathcal{E})$$

With this choice, we find

$$\int \hat{v}_2 f \, dv = \int \hat{v}_2 \left(v_2^2 + 2v_2\mathcal{B} + \mathcal{B}^2 \right) I_{(-\infty, 1)}(\mathcal{E}) dv$$

Now $\qquad \mathcal{E} = \sqrt{1+|v|^2} + \mathcal{U} < 1 \Leftrightarrow \sqrt{1+|v|^2} < 1 - \mathcal{U}$

$$\Leftrightarrow 1+|v|^2 < (1-\mathcal{U})^2$$

$$\Leftrightarrow |v|^2 < (1-\mathcal{U})^2 - 1$$

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$$\Leftrightarrow |v| < \sqrt{(1-\mathcal{U})^2 - 1}$$

Note that if $\mathcal{U} \leq 0$, this is always defined. The integral becomes

$$\int_{|v| < \sqrt{(1-\mathcal{U})^2-1}} \hat{v_2}\left(v_2^2 + 2v_2\mathcal{B} + \mathcal{B}^2\right)dv$$

$$= 2\mathcal{B} \int_{|v| < \sqrt{(1-\mathcal{U})^2-1}} \frac{v_2^2}{\sqrt{1+|v|^2}}\, dv$$

$$= 2\pi\, \mathcal{B} \left[\frac{1}{3}(1-\mathcal{U})^3 - (1-\mathcal{U}) + \frac{2}{3}\right]$$

and the equation for $\mathcal{B}$ becomes

(A.4) $\qquad -\dfrac{1}{4\pi}\,\mathcal{B}''(x) = \dfrac{2\pi}{3}\Big[((1-\mathcal{U}(x))^3 - 3(1-\mathcal{U}(x)) + 2\Big]\mathcal{B}(x),$

when $\mathcal{U}(x) \leq 0$.

We will also need to evaluate $\int f dv$:

$$\int f dv = \int (v_2^2 + 2v_2\mathcal{B} + \mathcal{B}^2)I_{(-\infty,1)}(\mathcal{E})dv$$

$$= \int_{|v| < \sqrt{(1-\mathcal{U})^2-1}} (v_2^2 + \mathcal{B}^2)dv$$

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which gives

$$(A.5) \qquad \int f dv = \frac{\pi}{4}\left((1-\mathcal{U})^2-1\right)^2 + \pi \mathcal{B}^2\left((1-\mathcal{U})^2-1\right)$$

Next, we choose

$$\mathcal{U}(x) = \begin{cases} -A(1-x^2)^3 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where A is a parameter whose purpose will be explained shortly. Inserting this into (A.4), we have

$$(A.6) \qquad \mathcal{B}''(x) = -\frac{8\pi^2}{3}\sigma(x)\mathcal{B}(x) \quad \text{where}$$

$$\sigma(x) = \begin{cases} \left(1+A(1-x^2)^3\right)^3 - 3\left(1+A(1-x^2)^3\right) + 2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Comment: Different choices of g can simplify the equation for $\mathcal{B}$. For example, if $g = (C+\ell\mathcal{E})I_{(-\infty,1)}(\mathcal{E})$ where C is a constant chosen to force non-negativity of f, we obtain (using the same $\mathcal{U}$), $\mathcal{B}''(x) = \Sigma(x)$ where Σ is a 16th degree polynomial in x. This can be integrated twice to find a closed form solution $\mathcal{B}(x)$. It turned out, however, that this solution resulted in excessive amounts of charge present to maintain the steady state. Computation of the fields involves subtracting an integral of the

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background ion density from an integral of the electron density. Since these turned out to be large numbers, an extremely fine refinement of the mesh (resulting in a prohibitively large number of particles) was required to resolve the fields. The above choice of g was deemed the best after much experimentation. To compute a solution of (A.6), we use a modified shooting method. We choose (arbitrarily) β (1) =1 and require β' (1) = 0. We solve an initial value problem with this data specified, and vary the parameter A until we achieve a zero slope at x = 0. β is then extended as described at the beginning of this section.

The difference scheme used is

$$\frac{\beta^{k+1} - 2\beta^{k} + \beta^{k-1}}{(\Delta x)^2} = -\frac{8\pi^2}{3} \sigma^k(x)\beta^{k}$$

or $\qquad \beta^{k-1} = 2\beta^{k} - \beta^{k+1} - \frac{8\pi^2}{3} (\Delta x)^2 \sigma(x^k) \beta^{k}$

where $\beta^{n+1} = \beta^{n} = 1$, $\Delta x = \frac{1}{n}$, and we try to achieve $\beta^{-1} = \beta^{1}$

With n = 500, the value of A required to achieve the boundary conditions is found to be A = 0.808195. At this point, we consider β a known function. In practice, we have β at 500 mesh points in the interval [0,1]. In the program, where required, values of β at intermediate points are linearly interpolated, while values of B = β' are obtained using the Mean Value Theorem.

Having found β, we are able to determine n. For |x| < 1, we have from (A.2):

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$$n(x) = \int g\left(\sqrt{1+|v|^2} + \mathcal{U}, v_2 + \mathcal{B}\right)dv + \frac{1}{4\pi}\mathcal{U}''(x)$$

Using (A.5) and substituting in for $\mathcal{U}$ and $\mathcal{U}''$, we have

$$n(x) = \frac{\pi}{4}\left[\left(1+A(1-x^2)^3\right)^2 - 1\right]^2 + \pi\mathcal{B}^2(x)\left[\left(1+A(1-x^2)^3\right)^2 - 1\right]$$

$$- \frac{3A}{2\pi}(1-x^2)(5x^2-1) \quad \text{for } |x| < 1$$

and $\qquad n(x) = 0 \quad \text{for } |x| \geq 1$

Note also that the finite energy solution of the steady state RVM problem also solves the steady state version of the modified problem, (RVM~), with the condition $\tilde{E}_2 \equiv 0$. To show this, we write steady state (RVM~) in integral form:

$$\hat{v}_1\partial_x\tilde{f} + (\tilde{E} + \tilde{B}M\hat{v})\cdot\nabla_v\tilde{f} = 0$$

$$\tilde{E}_1(x) = 4\pi\int_{-\infty}^{x}\tilde{\rho}(y)dy$$

$$\tilde{E}_2(x) = -2\pi\int\tilde{j}_2(y)dy$$

$$\tilde{B}(x) = -2\pi\left[\int_{-\infty}^{x}\tilde{j}_2(y)dy - \int_{x}^{\infty}\tilde{j}_2(y)dy\right]$$

The requirement $\tilde{E}_2 = 0$ means

$$\tilde{B}(x) = -4\pi \int_{-\infty}^{x} \tilde{j}_2(y)\,dy$$

and the two problems are identical.

Appendix B: Rescaling to Achieve c=1

Suppose (f,E,B) is a solution of (RVM) for some value of c, i.e.

$$\partial_t f(t,x,v) + \hat{v}_1 \partial_x f(t,x,v) + (E_1(t,x) + c^{-1}\hat{v}_2 B(t,x))\partial_{v_1} f(t,x,v)$$

$$+ (E_2(t,x) - c^{-1}\hat{v}_1 B(t,x))\partial_{v_2} f(t,x,v) = 0$$

where
$$\rho(t,x) = \int f(t,x,v)dv - n(x)$$

$$j(t,x) = \int \hat{v} f(t,x,v)dv$$

$$\hat{v} = \frac{v}{\sqrt{1+c^{-2}|v|^2}}$$

$$E_1(t,x) = 4\pi \int_{-\infty}^{x} \rho(t,y)dy$$

$$\partial_t E_2(t,x) = -c\partial_x B(t,x) - 4\pi j_2(t,x)$$

$$\partial_t B = -c\partial_x E_2(t,x)$$

We re-scale as follows:

Let $\quad \bar{t} = t, \quad \bar{x} = cx, \quad \bar{v} = cv \quad$ and define:

$$\bar{f}(t,x,v) = c^2 f(\bar{t},\bar{x},\bar{v}) = c^2 f(t,cx,cv)$$

$$\bar{n}(x) = n(\bar{x}) = n(cx)$$

$$\bar{\rho}(t,x) = \int \bar{f}(t,x,v)dv - \bar{n}(x)$$

$$= \int c^2 f(t,cx,cv)dv - n(cx)$$

$$= \int c^2 f(t,cx,w)c^{-2}dw - n(cx)$$

$$= \int f(t,cx,w)dw - n(cx) = \rho(t,cx)$$

$$\bar{j}(t,x) = \int \hat{\hat{v}}\,\bar{f}(t,x,v)dv \quad \text{where} \quad \hat{\hat{v}} := \frac{v}{\sqrt{1+|v|^2}}$$

$$= \int \hat{\hat{v}}\, c^2 f(t,cx,cv)dv$$

$$= \int \hat{\hat{v}}c^2 f(t,cx,w)c^{-2}d\,w$$

Note that $(cv)^{\wedge} = \dfrac{cv}{\sqrt{1+c^{-2}c^2|v|^2}} = c\hat{\hat{v}},$

so $\hat{\hat{v}} = c^{-1}(cv)^{\wedge}$, and since $w = cv$,

$\hat{\hat{v}} = c^{-1}\hat{w}$ and we have

$$\bar{j}(t,x) = \int c^{-1}\hat{w}f(t,cx,w)dw$$

$$= c^{-1}j(t,cx)$$

Now define

$$\bar{E}_1(t,x) \;=\; 4\pi \int_{-\infty}^{x} \bar{\rho}(t,y)\,dy$$

$$=\; 4\pi \int_{-\infty}^{x} \rho(t,cy)\,dy$$

$$=\; 4\pi \int_{-\infty}^{cx} \rho(t,z)\,c^{-1}d\,z$$

$$=\; c^{-1}4\pi \int_{-\infty}^{cx} \rho(t,z)\,dz$$

$$=\; c^{-1}E_1(t,cx)$$

Similarly, define

$$\bar{E}_2(t,x) \;=\; c^{-1}E_2(t,cx)$$

$$\bar{B}(t,x) \;=\; c^{-1}B(t,cx)$$

Then ($\bar{f}, \bar{E}, \bar{B}$) solves (RVM) with $c = 1$. To show this, we first demonstrate that $\bar{E}_2$ and $\bar{B}$ are solutions of the one dimensional Maxwell System:

$$\partial_t \bar{E}_2(t,x) + \partial_x \bar{B}(t,x) = c^{-1}\left[\partial_t E_2(t,cx) + c\partial_x B(t,cx)\right]$$

$$= c^{-1}\left[-4\pi j_2(t,cx)\right]$$

$$= -4\pi c^{-1}\left[c\bar{j}_2(t,x)\right]$$

$$= -4\pi \bar{j}_2(t,x)$$

$$\partial_t \bar{B}(t,x) + \partial_x \bar{E}_2(t,x) = c^{-1}\left[\partial_t B(t,cx) + c\partial_x E_2(t,cx)\right]$$

$$= 0$$

We also have $\partial_x \bar{E}_1(t,x) = 4\pi \bar{\rho}(t,x)$ by definition, and it remains to show that the Vlasov equation with $c = 1$ is satisfied. Toward this end, we first compute

$$\partial_t \bar{f}(t,x,v) = \partial_t\left[c^2 f(t,cx,cv)\right] = c^2 \partial_t f(t,cx,cv)$$

$$\partial_x \bar{f}(t,x,v) = \partial_x\left[c^2 f(t,cx,cv)\right] = c^3 \partial_x f(t,cx,cv)$$

$$\partial_{v_i} \bar{f}(t,x,v) = \partial_{v_i}\left[c^2 f(t,cx,cv)\right] = c^3 \partial_{v_i} f(t,cx,cv), \qquad i = 1, 2$$

$$\Rightarrow \quad \partial_t \bar{f}(t,x,v) + \hat{\bar{v}}_1 \partial_x \bar{f}(t,x,v) + \left(\bar{E}_1(t,x) + \hat{\bar{v}}_2 \bar{B}(t,x)\right) \partial_{v_1} \bar{f}(t,x,v)$$

$$+ \left(\bar{E}_2(t,x) - \hat{\bar{v}}_1 \bar{B}(t,x)\right) \partial_{v_2} \bar{f}(t,x,v)$$

$$= c^2 \partial_t f(t,cx,cv) + c^{-1}(cv_1)^\wedge c^3 \partial_x f(t,cx,cv)$$

$$+ \left(c^{-1} E_1(t,cx) + c^{-1}(cv_2)^\wedge c^{-1} B(t,cx)\right) c^3 \partial_{v_1} f(t,cx,cv)$$

$$+ \left(c^{-1} E_2(t,cx) - c^{-1}(cv_1)^\wedge c^{-1} B(t,cx)\right) c^3 \partial_{v_2} f(t,cx,cv)$$

$$= c^2 \partial_t f(t,cx,w) + c^2 \hat{w} \partial_x f(t,cx,w)$$

$$+ \left(c^{-1} E_1(t,cx) + c^{-2} \hat{w}_2 B(t,cx)\right) c^3 \partial_{w_1} f(t,cx,w)$$

$$+ \left(c^{-1} E_2(t,cx) - c^{-2} \hat{w}_1 B(t,cx)\right) c^3 \partial_{w_2} f(t,cx,w)$$

$$= c^2 \left\{ (\partial_t f + \hat{w} \partial_x f + [(E + c^{-1} B M \hat{w}) \cdot \nabla_w f)] \right\} \Big|_{(t,cx,w)}$$

$$= 0$$

Finally, we compute

$$\partial_t \bar{E}_1(t,x) = \partial_t [c^{-1} E_1(t,cx)]$$

$$= c^{-1} \partial_t E_1(t,cx)$$

$$= c^{-1}[-4\pi j_1(t,cx)]$$

$$= -4\pi[c^{-1}j_1(t,cx)]$$

$$= -4\pi \bar{j}_1(t,x), \quad \text{and as claimed,}$$

$$= (\bar{f}, \bar{E}, \bar{B}) \quad \text{solves} \quad (RVM) \quad \text{with} \quad c = 1.$$

We note here that the modified problem admits the same scaling. To see this, suppose $(\tilde{f}, \tilde{E}, \tilde{B})$ solves (RVM~) for some value of $c > 1$. Then the Vlasov equation and the expressions for $\tilde{\rho}$ and $\tilde{j}$ are the same as those in (RVM) with f, E and B replaced by $\tilde{f}$, $\tilde{E}$ and $\tilde{B}$. In (RVM~), the fields are given by

$$\tilde{E}_2(t,x) = \frac{1}{2}\left[\tilde{E}_2^0(x-ct) + \tilde{E}_2^0(x+ct) + \tilde{B}^0(x-ct) - \tilde{B}^0(x+ct)\right] -$$

$$\frac{2\pi}{c} \int \tilde{j}_2(t,y)dy$$

$$\tilde{B}(t,x) = \frac{1}{2}\left[\tilde{E}_2^0(x-ct) - \tilde{E}_2^0(x+ct) + \tilde{B}^0(x-ct) + \tilde{B}^0(x+ct)\right] -$$

$$\frac{2\pi}{c} \int_{-\infty}^{x} \tilde{j}_2(t,y)dy + \frac{2\pi}{c} \int_{x}^{\infty} \tilde{j}_2(t,y)dy$$

We define the following:

$$\tilde{\bar{f}}(t,x,v) \; := \; c^2 f(t,cx,cv)$$

$$\tilde{\bar{n}}(x) \; := \; \tilde{n}(cx)$$

$$\tilde{\bar{\rho}}(t,x) \; := \; \tilde{\rho}(t,cx)$$

$$\tilde{\bar{j}}(t,x) \; := \; c^{-1}\tilde{j}(t,cx)$$

$$\tilde{\bar{E}}(t,x) \; := \; c^{-1}\tilde{E}(t,cx)$$

$$\tilde{\bar{B}}(t,x) \; := \; c^{-1}\tilde{B}(t,cx)$$

With these, we find

$$\tilde{\bar{E}}_2(t,x) \; = \; c^{-1}\tilde{E}_2(t,cx)$$

$$= \frac{c^{-1}}{2}\left[\tilde{E}_2^0(cx-ct) + \tilde{E}_2^0(cx+ct) + \tilde{B}^0(cx-ct) - \tilde{B}^0(cx+ct)\right] -$$

$$\frac{2\pi}{c^2}\int \tilde{j}_2(t,y)dy$$

Since $\tilde{\bar{E}}_2^0(x) = \tilde{\bar{E}}_2(0,x) = c^{-1}\tilde{E}_2(0,cx) = c^{-1}\tilde{E}_2^0(cx)$, and a similar result

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holds for $\bar{\tilde{B}}(x)$, we have

$$\bar{\tilde{E}}_2(t,x) = \frac{1}{2}\left[c^{-1}\tilde{E}_2^0(c(x-t)) + c^{-1}\tilde{E}_2^0(c(x+t)) + c^{-1}\tilde{B}^0(c(x-t)) - c^{-1}\tilde{B}^0(c(x+t))\right]$$

$$- \frac{2\pi}{c}\int c^{-1}\tilde{j}_2(t,y)dy$$

$$= \frac{1}{2}\left[\bar{\tilde{E}}_2^0(x-t) + \bar{\tilde{E}}_2^0(x+t) + \bar{\tilde{B}}^0(x-t) - \bar{\tilde{B}}^0(x+t)\right] - \frac{2\pi}{c}\int c^{-1}\tilde{j}_2(t,cz)cdz$$

$$= \frac{1}{2}\left[\bar{\tilde{E}}_2^0(x-t) + \bar{\tilde{E}}_2^0(x+t) + \bar{\tilde{B}}^0(x-t) - \bar{\tilde{B}}^0(x+t)\right] - 2\pi\int \tilde{j}_2(t,y)dy$$

Similarly,

$$\bar{\tilde{B}}(t,x) = \frac{1}{2}\left[c^{-1}\tilde{E}_2^0(c(x-t)) - c^{-1}\tilde{E}_2^0(c(x+t)) + c^{-1}\tilde{B}^0(c(x-t)) + c^{-1}\tilde{B}^0(c(x-t))\right]$$

$$- \frac{2\pi}{c}\int_{-\infty}^{cx} c^{-1}\tilde{j}_2(t,y)dy + \frac{2\pi}{c}\int_{cx}^{\infty} c^{-1}\tilde{j}_2(t,y)dy$$

$$= \frac{1}{2}\left[\bar{\tilde{E}}_2^0(x-t) - \bar{\tilde{E}}_2^0(x+t) + \bar{\tilde{B}}^0(x-t) + \bar{\tilde{B}}^0(x+t)\right]$$

$$- 2\pi\int_{-\infty}^{x} c^{-1}\tilde{j}_2(t,cz)dz + 2\pi\int_{x}^{\infty} c^{-1}\tilde{j}_2(t,czx)dz$$

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$$= \frac{1}{2} \left[\bar{\tilde{E}}_2^0(x-t) - \bar{\tilde{E}}_2^0(x+t) + \bar{\tilde{B}}^0(x-t) + \bar{\tilde{B}}^0(x+t) \right]$$

$$- 2\pi \int_{-\infty}^{x} \bar{\tilde{j}}_2(t,y)dy + 2\pi \int_{x}^{\infty} \bar{\tilde{j}}_2(t,y)dy$$

and so (RVM~) with $c = 1$ is satisfied.

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